Tabular Integration by Parts

A Jewel in the Rough

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As far as integration techniques go, the tabular method of integration by parts is underrated. In this presentation, I will discuss some of the less familiar applications and consequences of integration by parts and the tabular method.

<u>The Integration by Parts Formula</u>

$$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx$$

or

$$\int u\, dv = uv - \int v\, du$$

When using the integration by parts formula, one must choose u and dv wisely.

- u must be easy to differentiate.
- v must be easily obtained from dv.
- Ideally, $v\,du$ must be simpler than $u\,dv$.

Choosing u and dv wisely is very difficult for beginners.

As a general rule of thumb, our students could use the acronym LIATE.

L - Logarithmic

I - Inverse trigonometric

A - Algebraic

T - Trigonometric

E - Exponential

When using the integration by parts formula, choose u as the type of function that appears first in LIATE.

Example

Evaluate the indefinite integral $\int x^3 \ln x \, dx$.

According to LIATE we choose $u = \ln x$ and $dv = x^3 \, dx$.

$$\begin{aligned} \int x^3 \ln x \, dx &= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^3 \, dx \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C \end{aligned}$$

Instead of LIATE, some people prefer to use LIPTE, where the P stands for polynomial.

Application - Exact DEs

To solve exact differential equations, we integrate, by parts when necessary.

Example

$$2xy\,dx + (x^2 - 1)\,dy = 0$$

$$\left(\int 2xy\,dx\right) + \left(\int x^2\,dy - \int 1\,dy\right) = C$$

$$\left(x^2y - \int x^2\,dy\right) + \left(\int x^2\,dy - y\right) = C$$

$$x^2y - y = C$$

Application - Linearizations

$$f(x) = f(a) + \int_a^x f'(t)\,dt$$

$$u = f'(t), \qquad dv = dt$$

$$du = f''(t)\,dt, \qquad v = -(x-t)$$

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x (x-t)f''(t)\,dt$$

$$f(x) = f(a) + f'(a)(x-a) + f''(z)\frac{(x-a)}{2}$$

The Tabular Method

$$\int x^3 \ln x \, dx = \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^3 \, dx$$

signs	u and du/dx	dv/dx and $\int dv$
$+$	$\ln x$	x^3
$-$	$\frac{1}{x}$	$\frac{1}{4}x^4$

Of course, the tabular method is more useful for problems requiring repeated integration by parts.

Example

Use the tabular method to evaluate $\int x^3 \cos x \, dx$.

signs	u and du/dx	dv/dx and $\int dv$
$+$	x^3	$\cos x$
$-$	$3x^2$	$\sin x$
$+$	$6x$	$-\cos x$
$-$	6	$-\sin x$
$+$	0	$\cos x$

$$\int x^3 \cos x \, dx = x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + C$$

Repeated integration by parts and the tabular method offer an elementary approach to a number of more advanced topics.

Consider the definite integral $\int_0^1 e^{-x}\,dx$.

signs	u and du/dx	dv/dx and $\int dv$
$+$	e^{-x}	1
$-$	$-e^{-x}$	x
$+$	e^{-x}	$x^2/2$
$-$	$-e^{-x}$	$x^3/6$
$+$	e^{-x}	$x^4/24$

Upon evaluating at 1 and 0, we get

$$\int_0^1 e^{-x}\,dx = \frac{1}{e} + \frac{1}{2e} + \frac{1}{6e} + \frac{1}{24e} + \int_0^1 \frac{x^4}{24} e^{-x}\,dx$$

e is irrational.

Proof: Suppose $e = a/b$ where a and b are positive integers and choose $n \geq \max\{b, a/b\}$.

$$\int_0^1 e^{-x}\, dx = 1 - \frac{1}{e} =$$

$$\frac{1}{e}\left(1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}\right) + \int_0^1 \frac{x^n}{n!} e^{-x}\, dx$$

Multiply by e and rearrange

$$e - 1 - \left(1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}\right) = e \int_0^1 \frac{x^n}{n!} e^{-x}\, dx$$

Multiply by $n!$

$$\text{Left side} = n!(e-1) - n!\left(1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}\right)$$
$$= \text{an integer}$$

$$\text{Right side} = e\int_0^1 x^n e^{-x}\,dx < \frac{e}{n+1}$$

In addition to showing that e is irrational, we've also shown that

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}\right).$$

We can use repeated integration by parts to find sums of other series. We'll come back to this idea.

Taylor's Polynomials

Using repeated integration by parts, we can generalize our notion of linearizations.

As before, we start with

$$f(x) = f(a) + \int_a^x f'(t)\,dt$$

$$\begin{array}{ccccc}
\underline{\text{signs}} & & \underline{u \text{ and } du/dx} & & \underline{dv/dx \text{ and } \int dv} \\
+ & \longrightarrow & f'(t) & & 1 \\
 & & & \searrow & \\
- & \longrightarrow & f''(t) & & -(x-t) \\
 & & & \searrow & \\
+ & \longrightarrow & f'''(t) & & (x-t)^2/2 \\
 & & & \searrow & \\
- & \longrightarrow & f^{(4)}(t) & & -(x-t)^3/6 \\
 & & & \searrow & \\
+ & \longrightarrow & f^{(5)}(t) & \longrightarrow & (x-t)^4/24
\end{array}$$

Upon evaluating at x and a, we get

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

$$+\frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \int_a^x \frac{f^{(5)}(t)}{4!}(x-t)^4\,dt$$

$$f(x) = \sum_{n=0}^{4} \frac{f^{(n)}(a)}{n!}(x-a)^n + \int_a^x \frac{f^{(5)}(t)}{4!}(x-t)^4\,dt$$

It follows that

$$f(x) = \sum_{n=0}^{4} \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(5)}(z)}{5!}(x-a)^5$$

where z is between a and x.

Of course, there was nothing special about 5. As long as f has enough continuous derivatives, we could continue the process.

This method of deriving Taylor polynomials (and proving Taylor's Theorem) is perhaps more natural than the traditional approach.

Tabular integration by parts can also be used to illustrate a number of other practical and theoretical results.

D. Horowitz, *Tabular Integration by Parts*, College Mathematics Journal, 21 (1990), pp. 307–311.

Available at the website *MAA Calculus Articles for Your Students*.

Modified Tabular Method

If the tabular method is modified, it can be used for a number of integrals that would not normally be considered "tabular type".

For example, consider the definite integral

$$\int_0^1 (1 - x^2)^3 \, dx.$$

$$
\begin{array}{lccc}
+ \longrightarrow & (1-x^2)^3 & & 1 \\
- \longrightarrow & 3(1-x^2)^2 & -2x & x \\
 & & & -2x^2 \\
+ \longrightarrow & 6(1-x^2) & -2x & \frac{-2x^3}{3} \\
 & & & \frac{4x^4}{3} \\
- \longrightarrow & 6 & -2x & \frac{4x^5}{15} \\
 & & & \frac{-8x^6}{15} \\
+ & 0 & & \frac{-8x^7}{105}
\end{array}
$$

Upon evaluating at 0 and 1, we get

$$\int_0^1 (1-x^2)^3\,dx = \frac{2^3 \cdot 3!}{1\cdot 3\cdot 5\cdot 7},$$

and it takes only a little leap of faith to conclude

$$\int_0^1 (1-x^2)^n\,dx = \frac{2^n \cdot n!}{1\cdot 3\cdot 5\cdots(2n+1)}.$$

Another good example for the modifed method is $\displaystyle\int_0^1 (x\ln x)^n\,dx = \frac{(-1)^n n!}{(n+1)^{n+1}}$.

Sums of Series

It is easy to see that

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{\ln 2}{2}.$$

However, if we start with

$$u = (1+x^2)^{-1} \text{ and } dv = x\,dx,$$

we can use the modified tabular method to show that

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} = \ln 2.$$

Instead of using repeated integration, we can streamline the approach.

Define the sequence $\{a_n\}$ where

$$a_n = \int_0^1 \frac{x^{2n-1}}{(1+x^2)^n}\,dx.$$

Upon integrating by parts, we have

$$a_n = \frac{1}{n2^{n+1}} + a_{n+1}.$$

Repeated use of this formula gives

$$\frac{\ln 2}{2} = \left(\sum_{n=1}^{K} \frac{1}{n2^{n+1}}\right) + a_{K+1}$$

and our result follows pretty easily from here.

Some other sums

$$\ln 2 = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{2^{n-1}n(n+1)(n+2)}$$

- Integral:

$$\int_0^1 \frac{x^2}{1+x}\,dx = \ln 2 - \frac{1}{2} = a_1$$

- Sequence:

$$a_n = \frac{2}{n(n+1)} \int_0^1 \frac{x^{n+1}}{(1+x)^n}\,dx$$

- Recurrence:

$$a_n = \frac{1}{2^{n-1}n(n+1)(n+2)} + a_{n+1}, \qquad n = 1, 2, 3, \ldots$$

$$\phi - 1 = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2 5^n}, \text{ where } \phi = (1+\sqrt{5})/2$$

- Integral:

$$\int_0^1 (4x+1)^{-3/2}\,dx = \frac{\phi - 1}{\sqrt{5}} = a_1$$

- Sequence:

$$a_n = \frac{(2n-1)!}{((n-1)!)^2} \int_0^1 \frac{x^{n-1}}{(4x+1)^{(2n+1)/2}}\,dx$$

- Recurrence:

$$a_n = \frac{(2n)!}{2(n!)^2 5^n \sqrt{5}} + a_{n+1}, \quad n = 1, 2, 3, \ldots$$

$$\frac{\pi}{2} = \sum_{n=0}^{\infty} \frac{n!}{(2n+1)!!}$$

- Integral:

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4} = \frac{1}{2} + a_1$$

- Sequence:

$$a_n = \frac{n!2^n}{(2n-1)!!} \int_0^1 \frac{x^{2n}}{(1+x^2)^{n+1}}\,dx$$

- Recurrence:

$$a_n = \frac{n!}{2(2n+1)!!} + a_{n+1}, \quad n = 1, 2, 3, \ldots$$

By using repeated integration by parts, infinite series can be introduced in an elementary way.

A good research problem for students is to use repeated integration to find an infinite series coinciding with a certain definite integral.

References

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