Math 240 - Test 1

February 13, 2025

Name key

Score

Show all work to receive full credit. Supply explanations where necessary. Give explicit solutions when possible. All integration must be done by hand (showing work), unless otherwise specified.

1. (12 points) Using our existence/uniqueness theorems, analyze each initial value problem to determine which one of these applies. **Do not attempt to solve the equations.**

   (A) A solution exists, but it is not guaranteed to be unique.

   (B) There is a unique solution.

   (C) A solution is not guaranteed to exist.

   Be sure to show work or explain.

   (a) $5x\dfrac{dy}{dx} = x^2 + \sqrt{8x - y^2}, \quad y(2) = 4$

   $$f(x,y) = \frac{x^2 + \sqrt{8x-y^2}}{5x}$$

   ANY RECTANGLE AROUND $(2,4)$ WILL CONTAIN POINTS WHERE $8x - y^2 < 0$.

   f IS NOT CONTINUOUS AROUND $(2,4)$.

   $\boxed{(C)}$

   (b) $x\dfrac{dy}{dx} + (x-7)y + x^2e^x = 0, \quad y(7) = 0$

   $$\frac{dy}{dx} + \frac{x-7}{x}y = -xe^x \leftarrow \text{LINEAR!}$$

   IT WILL HAVE A UNIQUE SOLUTION THROUGH ANY POINT WHERE $x \neq 0$.

   $\boxed{(B)}$

   (c) $\dfrac{dy}{dx} = \sqrt[3]{x-y}, \quad y(5) = 5$

   $f(x,y) = \sqrt[3]{x-y} \leftarrow$ CONTINUOUS EVERYWHERE

   $f_y(x,y) = \frac{1}{3}(x-y)^{-2/3}(-1) = \dfrac{-1}{3(x-y)^{2/3}} \leftarrow$ NOT CONT WHEN $x = y$.

   $\boxed{A}$

2. (12 points) A chemical reaction has order 3/2 if the rate of change of the amount of the reactant, A, satisfies the equation:

$$\frac{dA}{dt} = -kA^{3/2}, \quad A(0) = A_0.$$

(a) Solve the initial value problem to find a formula for $A(t)$.

$A^{-3/2}\, dA = -k\, dt$. IT MAKES PERFECT SENSE TO ASSUME $A > 0$.

$-2A^{-1/2} = -kt + C$

$\frac{2}{\sqrt{A}} = kt + C$

$A(0) = A_0 \Rightarrow \frac{2}{\sqrt{A_0}} = C$

$\frac{2}{\sqrt{A}} = kt + \frac{2}{\sqrt{A_0}}$

$\frac{1}{\sqrt{A}} = \frac{1}{2}kt + \frac{1}{\sqrt{A_0}}$

$$\boxed{A(t) = \frac{1}{\left(\frac{1}{2}kt + \frac{1}{\sqrt{A_0}}\right)^2}.}$$

(b) You are observing a chemical reaction that has order 3/2. At the start of the experiment (at $t = 0$), you had 0.40 g of substance. After 100 s, you have 0.15 g of the substance remaining. Find the constant k (round to the nearest hundredth) and write the complete formula for $A(t)$.

$A_0 = 0.4$

$\frac{1}{\sqrt{0.15}} = 50k + \frac{1}{\sqrt{0.4}} \Rightarrow k = \frac{\frac{1}{\sqrt{0.15}} - \frac{1}{\sqrt{0.4}}}{50}$

≈ 0.020

$$\boxed{A(t) = \frac{1}{\left(0.01t + \frac{1}{\sqrt{0.4}}\right)^2}}$$

(c) How long after you began the experiment did you have 0.20 g of substance?

$\frac{1}{\sqrt{0.2}} = 0.01t + \frac{1}{\sqrt{0.4}} \Rightarrow t = \frac{\frac{1}{\sqrt{0.2}} - \frac{1}{\sqrt{0.4}}}{0.01}$

$\approx \boxed{65.5 \text{ SEC}}$

3. (8 points) Use Euler's method with $h = 0.1$ to estimate $y(2.3)$.

$$\frac{dy}{dx} = x^2 + y + 1, \quad y(2) = 1$$

$f(x,y) = x^2 + y + 1$, $h = 0.1$

$y_{n+1} = y_n + h\, f(x_n, y_n)$

$y_0 = 1$

$x_0 = 2$

$y_1 = 1 + 0.1\left[(2)^2 + 1 + 1\right] = 1.6$

$x_1 = 2.1$

$y_2 = 1.6 + 0.1\left[(2.1)^2 + 1.6 + 1\right] = 2.301$

$x_2 = 2.2$

$y_3 = 2.301 + 0.1\left[(2.2)^2 + 2.301 + 1\right]$

$= 3.1151$

$x_3 = 2.3$

$\boxed{y(2.3) \approx 3.1151}$

4. (12 points) Find the general solution of the equation that appears in the previous problem:.

$$\frac{dy}{dx} = x^2 + y + 1.$$

(Do not worry about satisfying the initial condition from above.)

$\frac{dy}{dx} - y = x^2 + 1$

$\mu(x) = e^{\int -dx} = e^{-x}$

$e^{-x} y = \int (x^2+1)e^{-x}\, dx$

SIGNS	u AND DERIVS	dv/dx AND ANTIS
+	x^2+1	e^{-x}
−	$2x$	$-e^{-x}$
+	2	e^{-x}
−	0	$-e^{-x}$

$e^{-x} y = -(x^2+1)e^{-x} - 2xe^{-x} - 2e^{-x} + C$

$y(x) = -(x^2+1) - 2x - 2 + Ce^{x}$

$\boxed{y(x) = -x^2 - 2x - 3 + Ce^{x}}$

5. (12 points) Consider the equation $y' + \frac{4}{x}y = x^3y^2$.

(a) Find the general solution.

$y \equiv 0$ SING. SOL'N

BERNOULLI. DIVIDE BY y^2, ASSUMING $y \neq 0$

$$y^{-2}\frac{dy}{dx} + \frac{4}{x}y^{-1} = x^3$$

$$u = y^{-1} \Rightarrow \frac{du}{dx} = -y^{-2}\frac{dy}{dx}$$

$$-y^{-2}\frac{dy}{dx} - \frac{4}{x}y^{-1} = -x^3$$

$$\frac{du}{dx} - \frac{4}{x}u = -x^3$$

$$\mu(x) = e^{\int -\frac{4}{x}dx} = e^{-4\ln|x|} = \frac{1}{x^4}$$

$$\frac{1}{x^4}u = \int \frac{-x^3}{x^4}dx = \int -\frac{1}{x}dx$$

$$\frac{1}{x^4}u = -\ln|x| + C$$

$$u(x) = -x^4\ln|x| + Cx^4$$

$$\frac{1}{y(x)} = -x^4\ln|x| + Cx^4$$

$$\boxed{y(x) = \frac{1}{Cx^4 - x^4\ln|x|}}$$

(b) Show that your general solution cannot satisfy the initial condition $y(1) = 0$.

$$y(1) = \frac{1}{C - 0} = \frac{1}{C} \neq 0 \quad \text{FOR ANY } C.$$

THE RECIPROCAL OF A NUMBER CANNOT BE ZERO!

(c) Find a singular solution that satisfies $y(1) = 0$.

$$\boxed{y(x) \equiv 0}$$

6. (8 points) Rewrite the equation in differential form and show it is exact. Then solve the initial value problem.

$$\frac{dy}{dx} = \frac{xy^2 - 1}{1 - x^2y}, \quad y(0) = 1$$

$(1 - x^2y)\, dy = (xy^2 - 1)\, dx$

$\underbrace{-(xy^2 - 1)}_{M}\, dx + \underbrace{(1 - x^2y)}_{N}\, dy = 0$

$\frac{\partial M}{\partial y} = -2xy \quad = \quad \frac{\partial N}{\partial x} = -2xy$

EQUATION IS EXACT.

$\frac{\partial F}{\partial x} = 1 - xy^2 \Rightarrow F(x,y) = x - \frac{1}{2}x^2y^2 + g(y)$

$\frac{\partial F}{\partial y} = 1 - x^2y \Rightarrow F(x,y) = y - \frac{1}{2}x^2y^2 + h(x)$

SOLUTION:

$\boxed{x + y - \frac{1}{2}x^2y^2 = C \qquad y(0) = 1 \Rightarrow C = 1}$

7. (8 points) Listed below are some of the types of equations we have named, studied, and solved. Give the name of each class of equations.

(a) $\frac{dy}{dx} + P(x)y = Q(x)y^n$ BERNOULLI

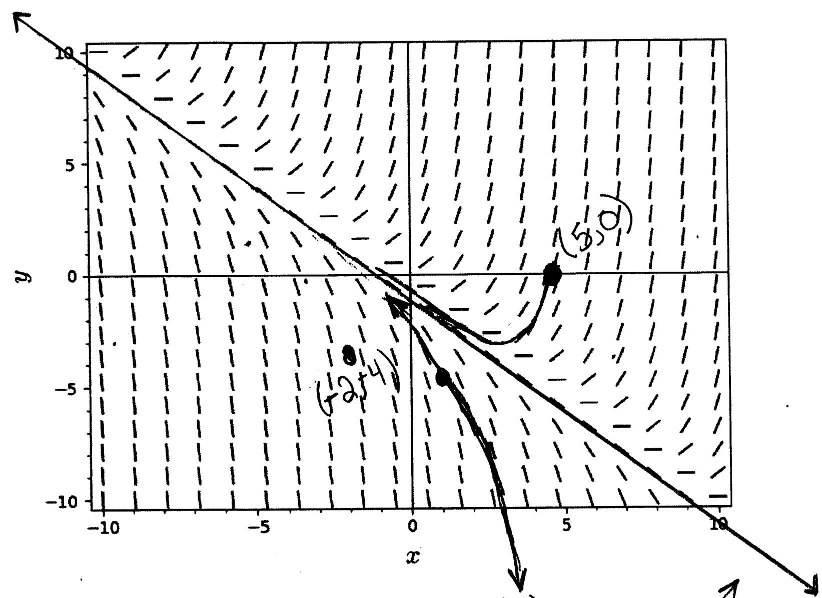
(b) $\frac{dy}{dx} = f(x)g(y)$ SEPARABLE

(c) $\frac{dy}{dx} = G\left(\frac{y}{x}\right)$ HOMOGENEOUS

(d) $\frac{dy}{dx} + p(x)y = q(x)$ 1ST. ORDER, LINEAR

(e) $\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$ EXACT

8. (10 points) A portion of the slope field for a differential equation is shown below.



(a) Based on the slope field, determine a particular solution for the differential equation:

$\boxed{y(x) = -x-1}$

(b) Give an initial point with a positive x-coordinate through which you would expect the solution to satisfy $\lim_{x\to\infty} = -\infty$. Roughly sketch the corresponding solution curve on the graph above.

(1, -5) SEE ABOVE.

(c) Consider the solution that satisfies $y(5) = 0$. Would you expect that solution to have relative maximum or minimum values? Explain. If so, estimate their x-coordinates.

A SINGLE MIN, SOMEWHERE AROUND X=3.

(d) Consider the solution that satisfies $y(-2) = -4$. Would you expect that solution to have relative maximum or minimum values? Explain.

NEITHER. LOOKS LIKE SOLUTION WOULD ALWAYS BE DECREASING.

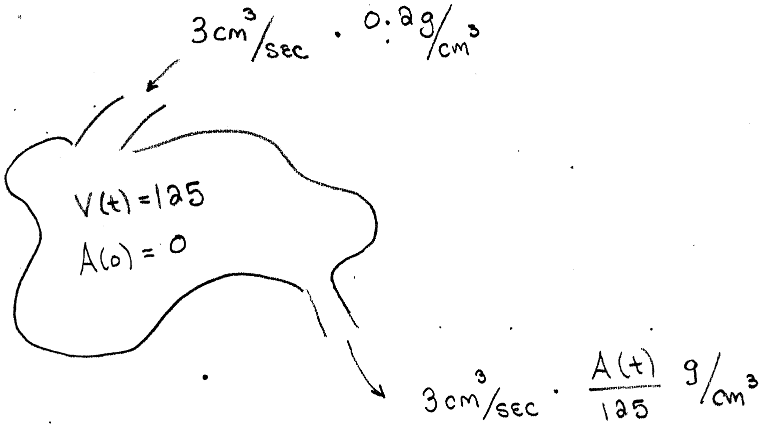
(e) Give a reason to argue that the given slope field cannot be for the equation $\underbrace{dy/dx}$ $= x^2 + y$.

$\frac{dy}{dx}$ WOULD BE POSITIVE EVERYWHERE IN QUAD II.

THAT'S NOT HAPPENING.

The following problems are take-home problems. They are due February 17. You must work on your own and show all work.

9. (6 points) Blood carries a drug into an organ at a rate of $3\,\text{cm}^3/\text{sec}$ and leaves at the same rate. The organ has a constant volume of $125\,\text{cm}^3$, and it initially contains none of the drug being introduced. If the concentration of the drug in the blood entering the organ is $0.2\,\text{g/cm}^3$, find the amount, $A(t)$, of drug in the organ at time t. When will the concentration of drug in the organ reach $0.1\,\text{g/cm}^3$?



$$\frac{dA}{dt} = 0.6 - \frac{3A}{125}, \quad A(0) = 0$$

$$\frac{dA}{dt} + \frac{3}{125}A = 0.6$$

$$\mu(t) = e^{\int \frac{3}{125}dt} = e^{3t/125}$$

$$e^{3t/125}A(t) = \int 0.6\,e^{3t/125}\,dt$$

$$e^{3t/125}A(t) = 25\,e^{3t/125} + C$$

$$A(t) = 25 + Ce^{-3t/125}$$

$$A(0) = 0 \Rightarrow C = -25$$

$$\boxed{A(t) = 25 - 25e^{-3t/125}}$$

$$\frac{A(t)}{125} = 0.1$$

$$A(t) = 12.5$$

$$\frac{1}{2} = e^{-3t/125}$$

$$t = \frac{-125\ln\left(\frac{1}{2}\right)}{3}$$

$$\approx \boxed{28.9\text{ sec}}$$

10. (6 points) Find the general solution: $xy'' - 3y' = 5x$

$u = y'$
$u' = y''$

$$xu' - 3u = 5x$$

$$u' - \frac{3}{x}u = 5$$

$$\mu(x) = e^{-\int \frac{3}{x}dx} = e^{-3\ln|x|} = \frac{1}{|x|^3} = \frac{1}{x^3}$$

LET'S ASSUME $x > 0$.

$$\frac{1}{x^3}u = \int \frac{5}{x^3}dx$$

$$= -\frac{5}{2}x^{-2} + C$$

$$u(x) = -\frac{5}{2}x + Cx^3 \Rightarrow y'(x) = -\frac{5}{2}x + Cx^3 \Rightarrow \boxed{y(x) = -\frac{5}{4}x^2 + Cx^4 + D}$$

11. (6 points) Here is an example of a problem involving a *pursuit curve*: Criminals are in a boat at the point $(1,0)$ when the police (at the origin) shine a spotlight on them. The criminals immediately evade the police by moving counter-clockwise at a 45° angle away from the light beam. Of course, the police will instantaneously readjust the light, and the criminals will continue to evade. The criminal's path will satisfy the initial value problem

$$\frac{dy}{dx} = \frac{y+x}{x-y}, \quad y(1) = 0.$$

Solve the initial value problem.

$$\frac{dy}{dx} = \frac{\frac{y}{x}+1}{1-\frac{y}{x}}$$

LET $u = \frac{y}{x}$ SO THAT

$ux = y$ AND $\frac{dy}{dx} = u + x\frac{du}{dx}$

$$u + x\frac{du}{dx} = \frac{u+1}{1-u}$$

$$x\frac{du}{dx} = \frac{u+1}{1-u} - \frac{u(1-u)}{1-u}$$

$$= \frac{1+u^2}{1-u}$$

$$\frac{1-u}{1+u^2}du = \frac{1}{x}dx$$

$w = 1+u^2$
$dw = 2u\,du, \quad \frac{1}{2}dw = u\,du$

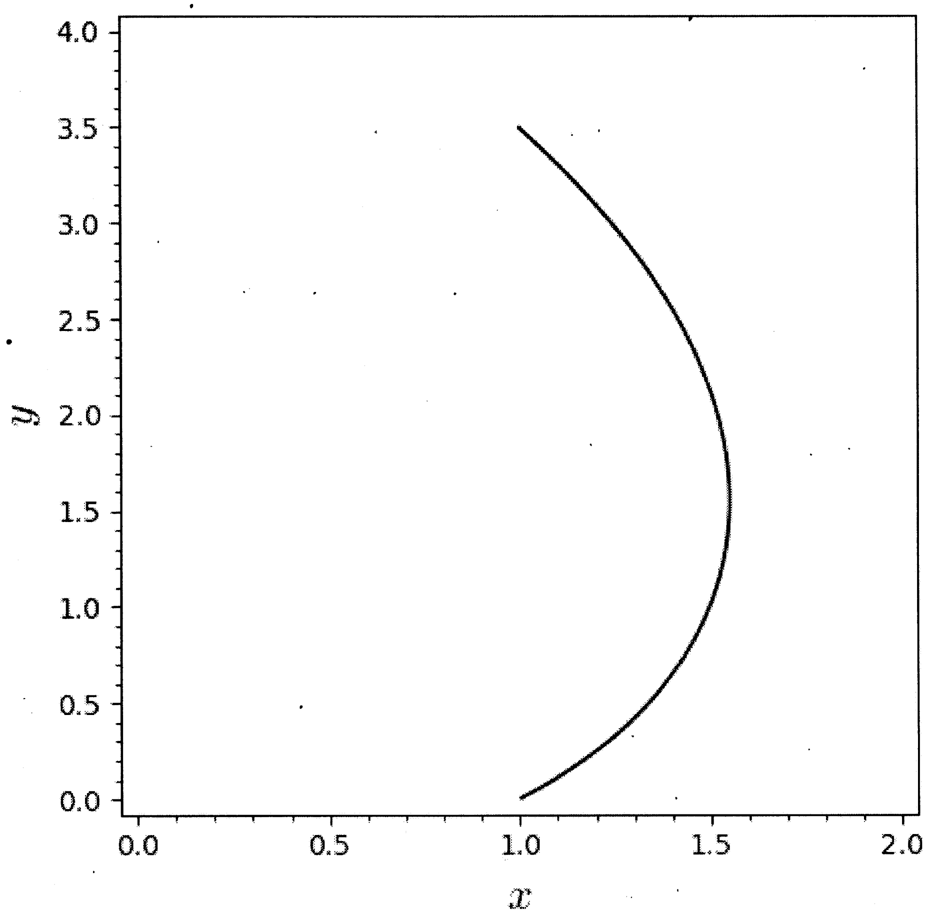
$$\left(\frac{1}{1+u^2} - \frac{u}{1+u^2}\right)du = \frac{1}{x}dx$$

$$\tan^{-1}u - \frac{1}{2}\ln(1+u^2) = \ln|x| + C$$

$$y(1) = 0 \Rightarrow u(1) = 0 \Rightarrow C = 0$$

$$\boxed{\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(1 + \frac{y^2}{x^2}\right) = \ln|x|}$$

FYI, THE GRAPH OF THE PATH IS ATTACHED.



4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
y
0.0
0.5
1.0
1.5
2.0
x