MTH 240 - Assignment # 9 key

1) $x^2y'' + 8xy' + 10y = 0$

(a) $y'' + \frac{8}{x}y' + \frac{10}{x^2}y = 0 \longrightarrow$ $x=0$ IS A SINGULAR PT.
HOWEVER, SINCE $p(x) = 8$ AND $q(x) = 10$ ARE ANALYTIC AT $x=0$, $x=0$ IS A REGULAR SINGULAR PT.

(b) $y = \sum_{n=0}^{\infty} a_n x^{n+s}$, $y' = \sum_{n=0}^{\infty} (n+s)a_n x^{n+s-1}$, $y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1)a_n x^{n+s-2}$

$$0 = \sum_{n=0}^{\infty} (n+s)(n+s-1)a_n x^{n+s} + \sum_{n=0}^{\infty} 8(n+s)a_n x^{n+s} + \sum_{n=0}^{\infty} 10a_n x^{n+s}$$

$$= \sum_{n=0}^{\infty} \left[(n+s)(n+s-1)a_n + 8(n+s)a_n + 10a_n\right] x^{n+s}$$

$$\Rightarrow (n+s)(n+s-1)a_n + 8(n+s)a_n + 10a_n = 0$$

$$\Rightarrow \left[(n+s)(n+s-1) + 8(n+s) + 10\right] a_n = 0$$

(c) ASSUME $a_0 \neq 0$ AND LET $n=0$.

THEN $s(s-1) + 8s + 10 = 0$ OR $s^2 + 7s + 10 = (s+2)(s+5) = 0$

$s = -2$, $s = -5$

(d) $s = -2$, $a_0 \neq 0 \Rightarrow \left[(n-2)(n-3) + 8(n-2) + 10\right] a_n = 0$; $n = 1, 2, 3, \ldots$

$\Rightarrow n^2 + 2n = 0$ OR $a_n = 0$; $n = 1, 2, 3, \ldots$

$n(n+2) = 0$ MUST HAVE $a_n = 0$; $n = 1, 2, 3, \ldots$

$n = 0$ OR $n = -2$

CONTINUED →

So, FOR $S=-2$, WE HAVE

$$a_0 \neq 0 \quad \text{AND} \quad a_n = 0;\ n=1,2,3,\ldots$$

$$\boxed{y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+s} = a_0 x^{-2}}$$

$$S=-5,\ a_0 \neq 0 \Rightarrow \left[(n-5)(n-6) + 8(n-5) + 10\right] a_n = 0;$$

$$n=1,2,3,\ldots$$

$$\Rightarrow n^2 - 3n \quad \text{OR} \quad a_n = 0;\ n=1,2,3,\ldots$$

$$n(n-3)=0$$

$$n=0,\ n=3$$

MUST HAVE

$a_n = 0$ FOR $n=1,2,4,5,\ldots$

$$y_2(x) = \sum_{n=0}^{\infty} a_n x^{n+s} = a_0 x^{-5} + a_3 x^{3-5}$$

$$= a_0 x^{-5} + a_3 x^{-2}$$

$$\boxed{y_2(x) = a_0 x^{-5} + a_3 x^{-2}}$$

(e) GENERAL SOLUTION $y(x) = \text{CONST} \times y_1(x) + \text{CONST} \times y_2(x)$

$$\boxed{y(x) = c_1 x^{-2} + c_2 x^{-5}}$$

(f) $x = e^t$ TRANSFORMS EQUATION TO

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 0$$

CHAR EQUATION $r^2 + 7r + 10 = (r+2)(r+5) = 0$

$$r = -2, \quad r = -5$$

$$y(t) = c_1 e^{-2t} + c_2 e^{-5t}$$

RESUB $x = e^t$

$$\boxed{y(x) = c_1 x^{-2} + c_2 x^{-5}}$$

2) $f(t) = \begin{cases} 2, & 0 \le t < 5 \\ 3, & t \ge 5 \end{cases}$

$$F(s) = \int_0^5 2e^{-st}\,dt + \int_3^\infty 3e^{-st}\,dt$$

$$= -\frac{2}{s}e^{-st}\Big|_{t=0}^{t=5} + -\frac{3}{s}e^{-st}\Big|_{t=3}^{t\to\infty}$$

$$= \frac{2}{s} - \frac{2}{s}e^{-5s} + \frac{3}{s}e^{-3s} - \underbrace{\lim_{t\to\infty}\frac{3}{s}e^{-st}}_{=0,\ s>0}$$

$$= \boxed{\frac{2}{s} - \frac{2}{s}e^{-5s} + \frac{3}{s}e^{-3s}, \quad s > 0}$$

3) $f(t) = \begin{cases} t+1, & 0 \le t \le 2 \\ 0, & t > 2 \end{cases}$

$$F(s) = \int_0^2 (t+1)e^{-st}\,dt = -\frac{(t+1)}{s}e^{-st} - \frac{1}{s^2}e^{-st}\Big|_{t=0}^{t=2}$$

+	$t+1$	e^{-st}
−	1	$-\frac{1}{s}e^{-st}$
+	0	$\frac{1}{s^2}e^{-st}$

$$= \boxed{-\frac{3}{s}e^{-2s} - \frac{1}{s^2}e^{-2s} + \frac{1}{s} + \frac{1}{s^2}}$$

4) $f(t) = \sin t\,,\ t \geq 0$

$$|\sin t| \leq 1e^{0t} = 1$$

$\sin t$ IS OF EXPONENTIAL ORDER $\alpha = 0$.

5) $f(t) = e^{t^2},\ t \geq 0$

$$|e^{t^2}| > Me^{\alpha t}$$ FOR ANY M AND ANY α ONCE t IS LARGE.

e^{t^2} IS NOT OF EXPONENTIAL ORDER.

6) (a) $F(s) = \dfrac{3s}{s^2+4} + \dfrac{1}{2}\,\dfrac{2}{s^2+4}$

$$\boxed{f(t) = 3\cos 2t + \frac{1}{2}\sin 2t}$$

(b) $F(s) = \dfrac{5}{s}e^{-3s}$

$$\boxed{f(t) = 5\,\mathcal{U}(t-3) = \begin{cases} 0, & 0 \leq t < 3 \\ 5, & t \geq 3 \end{cases}}$$

(c) $F(s) = \dfrac{3}{s} - \dfrac{2}{s^4} - \dfrac{8}{6-s} = \dfrac{3}{s} - \dfrac{1}{3}\cdot\dfrac{3!}{s^4} + \dfrac{8}{s-6}$

$$\boxed{f(t) = 3 - \frac{1}{3}t^3 + 8e^{6t}}$$