MTH 240 Assignment 4 Key

1) $y'' = y' e^y$; $y(0) = 0$, $y'(0) = 1$

$u = y'$

$y'' = \frac{du}{dy}\frac{dy}{dx} = u\frac{du}{dy}$

$u\frac{du}{dy} = u e^y$

$\frac{du}{dy} = e^y$, ASSUMING $u \neq 0$

$du = e^y\,dy$

$u = e^y + C$

WHEN $x = 0$, $y = 0$ & $u = 1$

THEREFORE $1 = e^0 + C \Rightarrow C = 0$

$u = e^y \Rightarrow \frac{dy}{dx} = e^y \Rightarrow e^{-y}\,dy = dx$

$-e^{-y} = x + D$

$y(0) = 0 \Rightarrow D = -1$

$e^{-y} = 1 - x$

$-y = \ln(1-x)$

$\boxed{y(x) = -\ln(1-x)}$

2) $y'' - \frac{4}{x}y' + \frac{6}{x^2}y = 0; \quad y(0) = 0, \; y'(0) = 0$

SHOULD NOT EXPECT A UNIQUE SOLUTION THROUGH $X=0$ BECAUSE THE COEFFICIENT FUNCTIONS ARE NOT CONTINUOUS AT $X=0$.

OUR EXISTENCE THEOREM DOES NOT APPLY.

3) $y_1(x) = x^2 + 1$

$y_2(x) = x^2 + 3x$

$y_3(x) = 1 - 3x$

SINCE $y_3(x) = y_1(x) - y_2(x)$,

y_3 IS A LINEAR COMBO OF y_1 & y_2.

THE FUNCTIONS ARE DEPENDENT.

4) $(x-1)y'' - xy' + y = 0$

a)
$$\left.\begin{aligned} y_1(x) &= x \\ y_1'(x) &= 1 \\ y_1''(x) &= 0 \end{aligned}\right\} \quad (x-1)(0) - x(1) + x = -x + x = 0 \checkmark$$

$$\left.\begin{aligned} y_2(x) &= e^x \\ y_2'(x) &= e^x \\ y_2''(x) &= e^x \end{aligned}\right\} \quad (x-1)e^x - xe^x + e^x = xe^x - e^x - xe^x + e^x$$
$$= 0 \checkmark$$

b)
$$W = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = xe^x - e^x = e^x(x-1) \neq 0$$
AS LONG AS $x \neq 1$

c) $(x-1)y'' - xy' + y = 2x - x^2$

$$\left.\begin{aligned} y_p(x) &= 2 + x^2 \\ y_p'(x) &= 2x \\ y_p''(x) &= 2 \end{aligned}\right\} \quad \begin{aligned} &(x-1)(2) - x(2x) + (2+x^2) \\ &= 2x - 2 - 2x^2 + 2 + x^2 = 2x - x^2 \checkmark \end{aligned}$$

d) $(x-1)y'' - xy' + y = \underbrace{2x^2 - 4x}_{-2(2x - x^2)}$

GEN. SOLUTION

IS $y(x) = -2(2+x^2) + c_1 x + c_2 e^x$

CONTINUED →

PART d CONTINUED...

$$y(x) = -4 - 2x^2 + c_1 x + c_2 e^x$$

$$y(0) = 3 \Rightarrow -4 + c_2 = 3 \Rightarrow c_2 = 7$$

$$y'(0) = 9 \Rightarrow c_1 + c_2 = 9 \Rightarrow c_1 = 2$$

$$\boxed{y(x) = -4 - 2x^2 + 2x + 7e^x}$$

e) $$\underbrace{y'' - \frac{x}{x-1}y' + \frac{1}{x-1}y}\; = \frac{2x - x^2}{x-1} \;;\; y(0) = 3,\; y'(0) = 9$$

THE COEFFICIENTS AND RHS ARE CONTINUOUS EVERYWHERE EXCEPT $x = 1$.

AS LONG AS WE STAY AWAY FROM $x = 1$, OUR THEOREM GUARANTEES A UNIQUE SOLUTION. So, YES!

5) $y y'' + (y')^2 = 0$

a) $\left.\begin{array}{l} y_1(x) = 1 \\ y_1'(x) = 0 \\ y_1''(x) = 0 \end{array}\right\}$ $(1)(0) + (0)^2 = 0$ ✓

$\left.\begin{array}{l} y_2(x) = \sqrt{x} \\ y_2'(x) = \dfrac{1}{2\sqrt{x}} \\ y_2''(x) = \dfrac{-1}{4\sqrt{x^3}} \end{array}\right\}$ $(\sqrt{x})\left(-\dfrac{1}{4\sqrt{x^3}}\right) + \left(\dfrac{1}{2\sqrt{x}}\right)^2$

$= -\dfrac{1}{4x} + \dfrac{1}{4x} = 0$ ✓

b) LET $y_3(x) = y_1(x) + y_2(x) = 1 + \sqrt{x}$

$y_3(x)$ IS NOT A SOLUTION!

$\left.\begin{array}{l} y_3(x) = 1 + \sqrt{x} \\ y_3'(x) = \dfrac{1}{2\sqrt{x}} \\ y_3''(x) = -\dfrac{1}{4\sqrt{x^3}} \end{array}\right\}$ $(1+\sqrt{x})\left(-\dfrac{1}{4\sqrt{x^3}}\right) + \left(\dfrac{1}{2\sqrt{x}}\right)^2$

$= -\dfrac{1}{4\sqrt{x^3}} - \dfrac{1}{4x} + \dfrac{1}{4x} = -\dfrac{1}{4\sqrt{x^3}} \neq 0$

c) $y y'' + (y')^2 = 0$

THIS EQUATION IS "HOMOGENEOUS", BUT

IT IS <u>NOT LINEAR</u>.

WE SHOULD NOT EXPECT A SUM OF

SOLUTIONS TO BE A SOLUTION.

6) $6y'' - 7y' - 3y = 0$

CHAR EQN : $6r^2 - 7r - 3 = 0$

$$6r^2 - 9r + 2r - 3 = 0$$

$$3r(2r-3) + 1(2r-3) = 0$$

$$(3r+1)(2r-3) = 0$$

$$r = -\frac{1}{3}, \quad y = \frac{3}{2}$$

$$\boxed{y(x) = c_1 e^{-x/3} + c_2 e^{3x/2}}$$

7) $9y'' - 12y' + 4y = 0$

CHAR EQN : $9r^2 - 12r + 4 = 0$

$$(3r-2)^2 = 0$$

$$r = 2/3, \quad r = 2/3$$

$$\boxed{y(x) = c_1 e^{2x/3} + c_2 x e^{2x/3}}$$

8) $y''' - 2y'' - y' + 2y = 0; \quad y(0) = 2, \ y'(0) = 3, \ y''(0) = 5$

CHAR EQN : $r^3 - 2r^2 - r + 2 = 0$

$$r^2(r-2) - 1(r-2) = 0$$

$$(r^2 - 1)(r-2) = 0$$

$$r = 1, \ r = -1, \ r = 2$$

$y(x) = c_1 e^x + c_2 e^{-x} + c_3 e^{2x}$

$y'(x) = c_1 e^x - c_2 e^{-x} + 2c_3 e^{2x}$

$y''(x) = c_1 e^x + c_2 e^{-x} + 4c_3 e^{2x}$

$y(0) = 2 \Rightarrow c_1 + c_2 + c_3 = 2$

$y'(0) = 3 \Rightarrow c_1 - c_2 + 2c_3 = 3$

$y''(0) = 5 \Rightarrow c_1 + c_2 + 4c_3 = 5$

$2c_1 + 3c_3 = 5$

$2c_1 + 6c_3 = 8$

$3c_3 = 3$

$c_3 = 1$

$2c_1 + 3 = 5$

$c_1 = 1$

$c_1 + c_2 + c_3 = 2$

$c_2 = 0$

$$\boxed{y(x) = e^x + e^{2x}}$$

9) $y''' - y'' + 7y' = 0$

CHAR EQN: $r^3 - r^2 + 7r = 0$

$$r(r^2 - r + 7) = 0$$

$$r = 0, \quad r = \frac{1 \pm \sqrt{1 - 28}}{2} = \frac{1}{2} \pm \frac{3\sqrt{3}\, i}{2}$$

$$\alpha \pm \beta i$$

$y(x) = c_1 e^{0x} + c_2 e^{x/2} \cos\left(\frac{3\sqrt{3}}{2}x\right) + c_3 e^{x/2} \sin\left(\frac{3\sqrt{3}}{2}x\right)$

$$\boxed{y(x) = c_1 + e^{x/2}\left(c_2 \cos\left(\frac{3\sqrt{3}}{2}x\right) + c_3 \sin\left(\frac{3\sqrt{3}}{2}x\right)\right)}$$

10) $y'' - 2y' + 2y = 0; \quad y(\pi) = e^{\pi}, \quad y'(\pi) = 0$

CHAR EQN: $r^2 - 2r + 2 = 0$

$(r-1)^2 = -1$

$r = 1 \pm i \qquad \alpha = 1, \ \beta = 1$

$y(x) = c_1 e^x \cos x + c_2 e^x \sin x$

$y'(x) = c_1 e^x \cos x - c_1 e^x \sin x + c_2 e^x \sin x + c_2 e^x \cos x$

$y(\pi) = e^{\pi} \Rightarrow -c_1 e^{\pi} = e^{\pi} \Rightarrow c_1 = -1$

$y'(\pi) = 0 \Rightarrow e^{\pi} - 0 + 0 - c_2 e^{\pi} = 0 \Rightarrow c_2 = 1$

$$\boxed{y(x) = e^x \sin x - e^x \cos x}$$