Math 240 - Assignment 1

Name ____________________

January 22, 2026

Score __________

Show all work to receive full credit. Supply explanations when necessary. This assignment is due January 29.

---

1. Show (by substitution) that $y(x) = Cx^4e^{-2x}$ is a solution of the initial value problem

$$xy'' + (2x-3)y' + 2y = 0; \quad y(0) = 0,\ y'(0) = 0$$

for any constant C.

Solution

$$y' = C(4x^3 - 2x^4)e^{-2x} \quad \text{(Product rule and combine terms)}$$
$$y'' = C(12x^2 - 16x^3 + 4x^4)e^{-2x} \quad \text{(Product rule and combine terms)}$$
$$xy'' + (2x-3)y' + 2y =$$
$$C(12x^3 - 16x^4 + 4x^5)e^{-2x} + C(-12x^3 + 14x^4 - 4x^5)e^{-2x} + 2Cx^4e^{-2x}$$

Everything above cancels to make zero, so all that remains is to check the initial conditions.

$$y(0) = C(0)^4e^{-2(0)} = 0$$
$$y'(0) = C[4(0)^3 - 2(0)^4]e^{-2(0)} = 0$$

2. Let $y_1(t) = e^t$ and $y_2(t) = e^t \ln t$. Show (by substitution) that both y_1 and y_2 are solutions of

$$ty'' + (1-2t)y' + (t-1)y = 0.$$

Then show (by substitution) that $y(t) = \alpha y_1(t) + \beta y_2(t)$ is also solution for any constants α and β.

Solution

First, we show that y_1 is a solution.

$$y_1'(t) = e^t, \qquad y_1''(t) = e^t$$
$$ty_1'' + (1-2t)y_1' + (t-1)y_1 = te^t + (1-2t)e^t + (t-1)e^t = 0$$

Next, we show that y_2 is a solution.

$$y_2'(t) = e^t \ln t + e^t/t, \qquad y_2''(t) = e^t \ln t + 2e^t/t - e^t/t^2$$
$$ty_2'' + (1-2t)y_2' + (t-1)y_2 = t(e^t \ln t + 2e^t/t - e^t/t^2) + (1-2t)(e^t \ln t + e^t/t) + (t-1)e^t \ln t$$

$$= te^t \ln t + 2e^t - e^t/t + e^t \ln t + e^t/t - 2te^t \ln t - 2e^t + te^t \ln t - e^t \ln t = 0$$

Finally, we show that $y(t) = \alpha y_1(t) + \beta y_2(t) = e^t(\alpha + \beta \ln t)$ is a solution.

$$y'(t) = e^t(\alpha + \beta \ln t) + \beta e^t/t$$
$$y''(t) = e^t(\alpha + \beta \ln t) + 2\beta e^t/t - \beta e^t/t^2$$

$$ty'' + (1-2t)y' + (t-1)y$$
$$= t(e^t(\alpha+\beta \ln t)+2\beta e^t/t-\beta e^t/t^2)+(1-2t)(e^t(\alpha+\beta \ln t)+\beta e^t/t)+(t-1)e^t(\alpha+\beta \ln t)$$
$$= te^t(\alpha + \beta \ln t) + 2\beta e^t - \beta e^t/t + e^t(\alpha + \beta \ln t) + \beta e^t/t - 2te^t(\alpha + \beta \ln t) - 2\beta e^t +$$
$$te^t(\alpha + \beta \ln t) - e^t(\alpha + \beta \ln t) = 0$$

3. The following differential equation can be solved by straight-forward antidifferentiation. Find the general solution.
$$(1 + x^2)y' = \tan^{-1} x$$

Solution

$$\frac{dy}{dx} = \frac{\tan^{-1} x}{1 + x^2}$$
$$y(x) = \int \frac{\tan^{-1} x}{1 + x^2}\, dx$$
$$\text{Let } u = \tan^{-1} x, \text{ so that } du = \frac{1}{1 + x^2}\, dx.$$
$$y(x) = \int u\, du = \frac{u^2}{2} + C = \frac{(\tan^{-1})^2}{2} + C$$

4. The equation below is the *Black-Scholes-Merton equation*, which describes the price evolution of financial derivatives. (Black and Scholes received the 1997 Nobel Prize in Economics.) Classify the differential equation by saying whether it is ordinary or partial, linear or nonlinear. Also give its order and name the dependent and independent variables.
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S}$$

Solution

The equation is a 2nd-order, linear, partial differential equation with independent variables t and S and dependent variable V. The symbols r and σ represent constants.

5. Consider the equation $y' = 2xy + 1$. Is this differential equation linear or nonlinear? Explain how you know. Then verify (by substitution) that $y(x) = e^{x^2} \int_0^x e^{-t^2}\,dt$ is a solution.

Solution

The equation is linear. Once it is rewritten

$$y' - 2xy = 1,$$

it has the standard form of a 1st-order linear equation.

Now, by using the product rule and the fundamental theorem of calculus, we get

$$\begin{aligned} y'(x) &= \frac{d}{dx}\left[e^{x^2}\int_0^x e^{-t^2}\,dt\right] \\ &= 2xe^{x^2}\int_0^x e^{-t^2}\,dt + e^{x^2}e^{-x^2} \\ &= 2xy + 1. \end{aligned}$$

6. For which values of m is $y(x) = e^{mx}$ a solution of $y'' - 5y' + 4y = 0$?

Solution

Substitute y into the equation...

$$y'(x) = me^{mx}, \qquad y''(x) = m^2e^{mx}$$

$$0 = y'' - 5y' + 4y = m^2e^{mx} - 5me^{mx} + 4e^{mx} = (m^2 - 5m + 4)e^{mx}$$

The only way that the right-hand side can be zero is by having $(m^2 - 5m + 4) = (m-4)(m-1) = 0$. So, $m = 4$ or $m = 1$. It is easy to verify that $y_1(x) = e^{4x}$ and $y_2(x) = e^x$ are both solutions.

7. For a simple pendulum of fixed length ℓ, the acceleration of the pendulum bob is proportional to $\sin\theta$, where θ is the displacement angle from equilibrium. The acceleration is d^2s/dt^2, where $s = \ell\theta$. Write the differential equation for θ.

Solution

$\dfrac{d^2s}{dt^2} = k\sin\theta,$ where θ is some constant.

By differentiating $s = \ell\theta$ twice with respect to θ, we have

$$\frac{d^2s}{dt^2} = \ell\frac{d^2\theta}{dt^2}.$$

Therefore

$$\frac{d^2\theta}{dt^2} = \frac{k}{l}\sin\theta.$$

8. The graph of a function $y(x)$ rises in the first quadrant from the point $(0, 0)$ to the point (x, y). The area under the graph is one third of the area of the rectangle with opposite vertices at $(0, 0)$ and (x, y).

   (a) Briefly explain how the problem situation gives rise to the following equation:

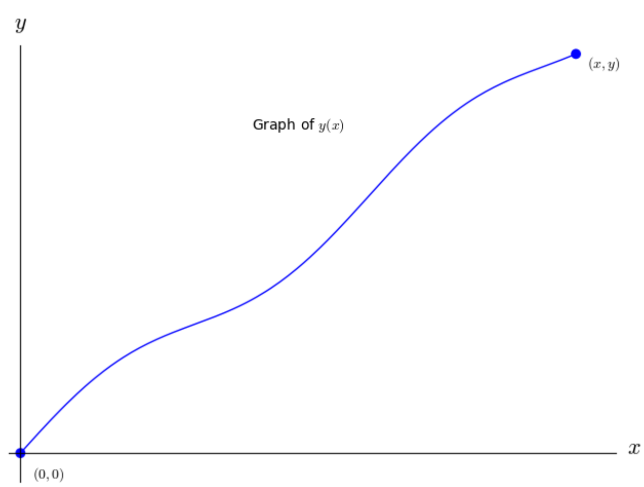
   $$\int_0^x y(t)\, dt = \frac{1}{3}xy.$$

   (b) By differentiating both sides of the equation, show that the following differential equation describes the function $y(x)$.

   $$y = \frac{1}{3}xy' + \frac{1}{3}y.$$

   <u>Solution</u>

   For part (a), refer to the graph below and use the idea that the definite integral gives the area under the curve. Also use that the area of a rectangle is its base length times its height.



   For part (b), differentiate both sides of the equation. The fundamental theorem of calculus gives $y(x)$ on the left, and the product rules gives

   $$\frac{1}{3}y + \frac{1}{3}xy'$$

   on the right.

9. Read the problem situation below. Write a differential equation having $y = g(x)$ as one of its solutions.

   The line normal to the graph of g at (x, y) passes through the point $(x/3, 1)$.

   <u>Solution</u>

The normal line passes through (x, y) and $(x/3, 1)$ so that its slope is given by

$$\frac{y-1}{x-x/3} = \frac{y-1}{2x/3} = \frac{3y-3}{2x}.$$

Since the normal line at (x, y) is perpendicular to the tangent line at (x, y), we must have

$$\frac{dy}{dx} = -\frac{2x}{3y-3}.$$

10. Solve the initial value problem: $\quad x(x^2-4)\dfrac{dy}{dx} = 1, \quad y(1) = 0.$

<u>Solution</u>

Rewrite the equation:

$$\frac{dy}{dx} = \frac{1}{x(x-2)(x+2)}.$$

Now antidifferentiate both sides to get

$$y(x) = \int \frac{1}{x(x-2)(x+2)}\,dx.$$

Compute the partial fraction decomposition of the integrand. (You can compute it mentally by using the cover-up method.)

$$\frac{1}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-2} = \frac{-1/4}{x} + \frac{1/8}{x-2} + \frac{1/8}{x+2}$$

It follows that

$$y(x) = \int \left(\frac{-1/4}{x} + \frac{1/8}{x-2} + \frac{1/8}{x+2}\right)dx = -\frac{1}{4}\ln|x| + \frac{1}{8}\ln|x-2| + \frac{1}{8}\ln|x+2| + C.$$

Next we use the initial condition to determine C.

$$y(1) = 0 \Longrightarrow \frac{1}{8}\ln(3) + C = 0$$

$$C = -\frac{1}{8}\ln(3)$$