Math 240 - Final Exam A

May 9, 2025

Name Key

Score ____________

Show all work to receive full credit. You must work individually. This test is due May 15. All integration must be done by hand (showing work).

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1. (10 points) The homogeneous equation

$$ty'' + (1-2t)y' + (t-1)y = 0$$

has the two linearly independent solutions $y_1(t) = e^t$ and $y_2(t) = e^t \ln t$.

Use variation of parameters (with care!) to find the general solution of the initial value problem

$$ty'' + (1-2t)y' + (t-1)y = te^t; \quad y(1) = 3e,\ y'(1) = \frac{5}{2}e.$$

$$y'' + \frac{1-2t}{t}y' + \frac{t-1}{t}y = e^t$$

$$W = \begin{vmatrix} e^t & e^t \ln t \\ e^t & e^t \ln t + \frac{e^t}{t} \end{vmatrix} = \frac{e^{2t}}{t}$$

$$y_p(t) = y_1(t)v_1(t) + y_2(t)v_2(t)$$

WHERE

$$v_1(t) = \int \frac{-e^t \ln t\, e^t}{e^{2t}/t}\,dt = \int -t \ln t\,dt$$

$u = \ln t \quad dv = -t\,dt$

$du = \frac{1}{t}dt \quad v = -\frac{t^2}{2}$

$$= -\frac{t^2}{2}\ln t + \int \frac{t}{2}dt = -\frac{t^2}{2}\ln t + \frac{t^2}{4}$$

$$v_2(t) = \int \frac{e^t e^t}{e^{2t}/t}\,dt = \int t\,dt = \frac{1}{2}t^2$$

$$y_p(t) = e^t\left(-\frac{t^2}{2}\ln t + \frac{t^2}{4}\right) + e^t \ln t\left(\frac{1}{2}t^2\right) = \frac{t^2}{4}e^t$$

$$y(t) = c_1 e^t + c_2 e^t \ln t + \frac{t^2}{4}e^t$$

$$y(1) = 3e = c_1 e + \frac{1}{4}e$$

$$\Rightarrow c_1 = \frac{11}{4}$$

$$y'(t) = c_1 e^t + c_2 e^t \ln t + \frac{c_2 e^t}{t} + \frac{1}{2}te^t + \frac{t^2}{4}e^t$$

$$y'(1) = c_1 e + c_2 e + \frac{3}{4}e$$

$$= c_2 e + \frac{14}{4}e$$

$$y'(1) = \frac{5}{2}e \Rightarrow$$

$$c_2 = -1$$

$$\boxed{y(t) = \frac{11}{4}e^t - e^t \ln t + \frac{t^2}{4}e^t}$$

2. (10 points) Use Laplace transform methods to solve the following equation.

$$tx'' - tx' + x = 2, \quad x(0) = 2,\ x'(0) = -1$$

$$-\frac{d}{ds}\left(s^2 X(s) - 2s + 1\right) + \frac{d}{ds}\left(s X(s) - 2\right) + X(s) = \frac{2}{s}, \quad s > 0$$

$$-2s X(s) - s^2 X'(s) + 2 + X(s) + s X'(s) + X(s) = \frac{2}{s}$$

$$(-s^2 + s) X'(s) + (-2s + 2) X(s) = \frac{2}{s} - 2$$

$$(s^2 - s) X'(s) + (2s - 2) X(s) = 2 - \frac{2}{s} = \frac{2s-2}{s}$$

$$X'(s) + \frac{2}{s} X(s) = \frac{2}{s^2}$$

$$\mu(s) = e^{\int \frac{2}{s} ds} = e^{2 \ln |s|} = e^{\ln s^2} = s^2$$

$$s^2 X(s) = \int 2\, ds = 2s + C$$

$$X(s) = \frac{2}{s} + \frac{C}{s^2}$$

$$x(t) = 2 + Ct$$

$$x'(0) = -1 \Rightarrow C = -1$$

$$\boxed{x(t) = 2 - t}$$

3. (10 points) Solve the following one-dimensional heat equation with Dirichlet boundary conditions. Do not derive the solution method–just use the result we derived in class. (See Theorem 1 on page 593.)

$$\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t \geq 0,$$

$$u(0,t) = u(1,t) = 0,$$

$$u(x,0) = (1-x)x^2, \quad 0 \leq x \leq 1$$

$k = 5$
$L = 1$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n e^{-5\pi^2 n^2 t} \sin(n\pi x),} \quad \text{WHERE}$$

$$b_n = 2\int_0^1 (1-x)x^2 \sin(n\pi x)\,dx$$

$$= \frac{-4}{n^3\pi^3}\left[2(-1)^n + 1\right]$$

SIGNS	u AND DERIVS	dv/dx AND ANTI
$+$	$x^2 - x^3$	$\sin(n\pi x)$
$-$	$2x - 3x^2$	$-\frac{1}{n\pi}\cos(n\pi x)$
$+$	$2 - 6x$	$-\frac{1}{n^2\pi^2}\sin(n\pi x)$
$-$	-6	$\frac{1}{n^3\pi^3}\cos(n\pi x)$
$+$	0	$\frac{1}{n^4\pi^4}\sin(n\pi x)$

$$b_n = 2\left[\frac{-(x^2-x^3)}{n\pi}\cos(n\pi x) + \frac{(2x-3x^2)}{n^2\pi^2}\sin(n\pi x) + \frac{(2-6x)}{n^3\pi^3}\cos(n\pi x) + \frac{6}{n^4\pi^4}\sin(n\pi x)\right]_0^1$$

$$= 2\left[\frac{-4}{n^3\pi^3}(-1)^n - \frac{2}{n^3\pi^3}\right] = \frac{-4}{n^3\pi^3}\left(2(-1)^n + 1\right)$$