Math 240 - Test 1

February 13, 2025

Name key

Score

Show all work to receive full credit. Supply explanations where necessary. Give explicit solutions when possible. All integration must be done by hand (showing work), unless otherwise specified.

1. (12 points) Analyze each initial value problem to determine which one of these applies. **Do not attempt to solve the equations.**

   (A) A solution exists, but it is not guaranteed to be unique.

   (B) There is a unique solution.

   (C) A solution is not guaranteed to exist.

   Be sure to show work or explain.

   (a) $2x(y+1) = y\dfrac{dy}{dx}, \quad y(1) = 0$

   $\boxed{(C)}$

   $$\frac{dy}{dx} = \underbrace{\frac{2x(y+1)}{y}}$$

   $f(x,y) = \dfrac{2x(y+1)}{y}$ IS NOT CONT. ON A RECTANGLE AROUND $(1,0)$ SINCE f IS NOT DEFINED WHEN $y = 0$.

   (b) $\dfrac{dy}{dx} = \sqrt[3]{x - 2y}, \quad y(6) = 3$

   $f(x,y) = \sqrt[3]{x-2y}$, f IS CONT. EVERYWHERE.

   $\boxed{(A)}$

   $f_y(x,y) = \dfrac{1}{3}\dfrac{-2}{(x-2y)^{2/3}}$, f_y IS NOT CONTINUOUS ON A RECTANGLE AROUND $(6,3)$ SINCE IT IS NOT DEFINED WHEN $x = 2y$.

   (c) $x^2\dfrac{dy}{dx} + e^x y - \sin x = 0, \quad y(\pi) = 0$

   $$\frac{dy}{dx} + \frac{e^x}{x^2}y = \frac{\sin x}{x^2}$$

   THIS IS A LINEAR EQUATION WITH COEFFICIENT FUNCTIONS CONT. EVERYWHERE EXCEPT WHEN $x = 0$.

   $\boxed{(B)}$

   THERE IS DEFINITELY A UNIQUE SOL'N THROUGH ANY PT WHEN $x \neq 0$ AND, IN PARTICULAR, THROUGH $(\pi, 0)$.

2. (8 points) Consider the following differential equations labeled [A] and [B].

$$[A] \quad \frac{d^2x}{dt^2} = 3t^2 + \cos t \qquad\qquad [B] \quad \frac{dy}{dx} = 3y^2 + \cos x$$

(a) For equation [A], state the order of the equation, the independent variable, and the dependent variable. Also say whether the equation is linear or nonlinear.

2ND ORDER, INDEP. VAR. IS t, DEP. VAR. IS X.

EQUATION IS LINEAR.

(b) For equation [B], state the order of the equation, the independent variable, and the dependent variable. Also say whether the equation is linear or nonlinear.

1ST ORDER, INDEP. VAR. IS X, DEP. VAR. IS y.

EQUATION IS NONLINEAR (y^2).

(c) One of the equations is very easy to solve exactly, and the other is difficult or impossible. Which is which? Explain why the problems are so different.

[A] IS VERY EASY SINCE THE UNKNOWN FUNCTION X DOES NOT APPEAR ON THE RIGHT. SIMPLY INTEGRATE $(3t^2 + \cos t)$ TWICE TO GET $X(t)$.

[B] HAS THE UNKNOWN FUNCTION ON THE RIGHT. STRAIGHT FORWARD INTEGRATION IS NOT POSSIBLE. (AND THE EQUATION IS NOT LINEAR!)

3. (8 points) Use Euler's method with $h = 0.1$ to estimate $y(1.3)$.

$$\frac{dy}{dx} = xy + 1, \quad y(1) = 2$$

$y_{n+1} = y_n + h(x_n y_n + 1)$

$x_{n+1} = x_n + h$

$\Downarrow$

$y_1 = 2 + 0.1((1)(2)+1)$
$= 2.3$

$x_1 = 1.1$

$y_2 = 2.3 + 0.1((1.1)(2.3)+1)$
$= 2.653$

$x_2 = 1.2$

$y_3 = 2.653 + 0.1((1.2)(2.653)+1)$
$= 3.07136 \Rightarrow \boxed{y(1.3) \approx 3.07136}$

Follow-up: Without actually finding the exact solution, briefly describe a method for solving the equation.

1ST ORDER LINEAR. FORM INTEGRATING FACTOR

$\frac{dy}{dx} - xy = 1$ $\quad \mu(x) = e^{\int -x\,dx}$ AND THEN $\mu(x)\,y(x) = \int \mu(x)\,dx$.

4. (10 points) Solve the initial value problem: $xy' + 5y = 24x^3 + 21x^2, \quad y(-1) = 4$

$y' + \frac{5}{x} y = 24x^2 + 21x$ BASED ON INITIAL CONDITION, WE'LL ASSUME $X < 0$.

$\mu(x) = e^{\int \frac{5}{x} dx} = e^{5 \ln|x|} = |X|^5 = -X^5, \quad X < 0$

$-x^5 y = \int (-x^5)(24x^2 + 21x)\, dx = \int (-24x^7 - 21x^6)\, dx$

$x^5 y = 3x^8 + 3x^7 + C$

$y(x) = 3x^3 + 3x^2 + \frac{C}{x^5}$

$y(-1) = 4 \Rightarrow 4 = (-3) + 3 - C$

$C = -4$

$y(x) = 3x^3 + 3x^2 - \frac{4}{x^5}$

5. (10 points) Show that the equation is exact. Then solve the initial value problem.

$$\underbrace{3x^2(1 + \ln y)}_{M}\, dx + \underbrace{\left(\frac{x^3}{y} - 2y\right)}_{N} dy = 0, \quad y(2) = 1$$

← NOTICE THAT $0 < y < \infty$ BECAUSE OF $\ln y$.

$\frac{\partial M}{\partial y} = \frac{3x^2}{y} = \frac{\partial N}{\partial x} = \frac{3x^2}{y} \Rightarrow$ EQUATION IS EXACT.

$F_x(x,y) = 3x^2(1 + \ln y) \Rightarrow F(x,y) = x^3(1 + \ln y) + g(y)$

$F_y(x,y) = \frac{x^3}{y} - 2y \Rightarrow F(x,y) = x^3 \ln y - y^2 + h(x), \quad y > 0$

$F(x,y) = x^3 \ln y + x^3 - y^2$

SOLUTION IS

$x^3 \ln y + x^3 - y^2 = C$

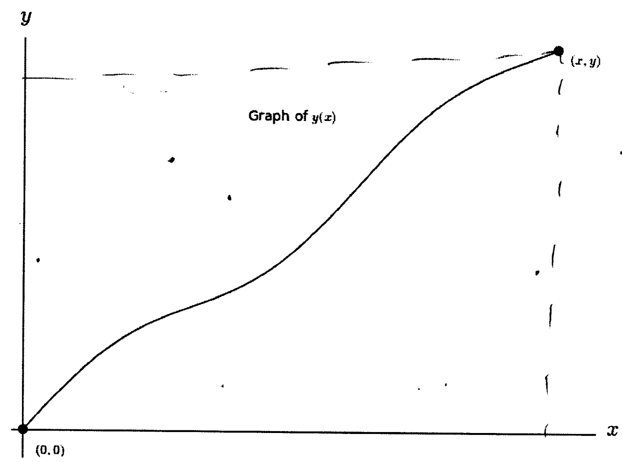
$C = F(2,1) = 7$

$x^3 \ln y + x^3 - y^2 = 7$

OR

$x = \sqrt[3]{\frac{7 + y^2}{1 + \ln y}}$

6. (12 points) The graph of a function $y(x)$ rises in the first quadrant from the point $(0,0)$ to the point (x,y). The area under the graph is one third of the area of the rectangle with opposite vertices at $(0,0)$ and (x,y).



(a) Briefly explain how the problem situation gives rise to the following equation:

$$\int_0^x y(t)\,dt = \frac{1}{3}xy.$$

$\int_0^x y(t)\,dt$ = AREA UNDER THE CURVE, AND $X \cdot Y$ (BASE x HEIGHT) IS THE AREA OF THE RECT.

(b) Differentiating both sides of the equation gives the differential equation

$$y = \frac{1}{3}xy' + \frac{1}{3}y.$$

Solve the differential equation.

$$3y = xy' + y \Rightarrow xy' = 2y \Rightarrow \frac{dy}{y} = \frac{2}{x}dx$$

$$e^{\ln|y|} = e^{2\ln|x| + C_1}$$

$$|y| = C_2 |x|^2 \Rightarrow \boxed{y(x) = C_3 x^2}$$

(c) Argue that the differential equation has infinitely many solutions satisfying $y(0) = 0$. (This is contrary to what you might expect from our existence/uniqueness theorems.)

FOR ANY NUMBER C_3, $y(x) = C_3 x^2$ IS A SOLUTION THAT SATISFIES $y(0) = 0$.

(BASED ON OUR EXISTENCE THEOREM, WE SHOULD NOT EXPECT A SOLUTION AT ALL!)

7. (10 points) Say whether the equation is *separable*, *linear*, or *exact*. Then solve the initial value problem.

$$2x(y+1) = y\frac{dy}{dx}, \quad y(1) = 0$$

SEPARABLE

$$2x\,dx = \frac{y}{y+1}\,dy$$

$$\int 2x\,dx = \int \frac{y}{y+1}\,dy$$

$$= \int \frac{y+1-1}{y+1}\,dy$$

$$= \int \left(1 - \frac{1}{y+1}\right) dy$$

$$x^2 + C = y - \ln|y+1|$$

ASSUME $y > -1$ BASED ON IC

$$x^2 + C = y - \ln(y+1)$$

$$y(1) = 0 \Rightarrow 1 + C = 0 \Rightarrow C = -1$$

$$\boxed{x^2 - 1 = y - \ln(y+1)}$$

Follow-up: Briefly explain why you may not have initially expected a solution to exist.

$$\frac{dy}{dx} = \frac{2x(y+1)}{y} \leftarrow$$

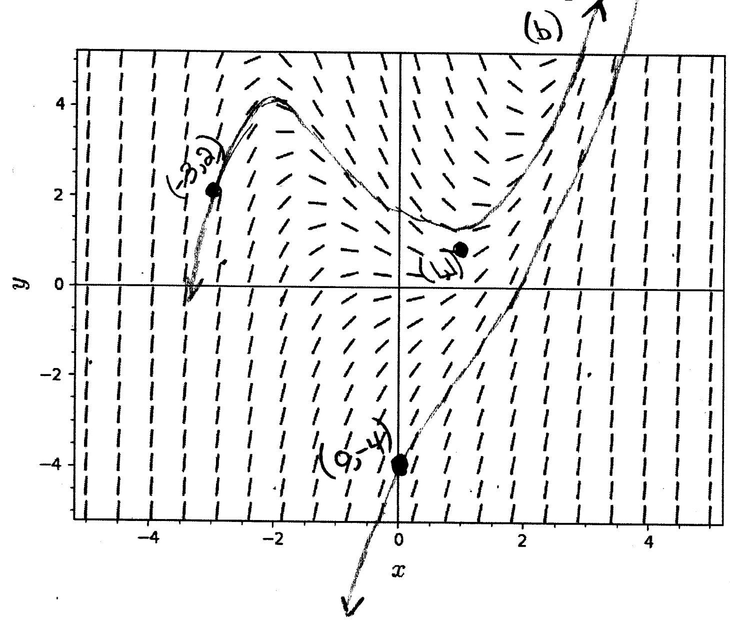
NOT CONTINUOUS ON RECTANGLE AROUND $(1,0)$.

SEE PROBLEM 1(a).

8. (4 points) Look back through the test at your solutions of differential equations. Give an example of an implicit solution and say why it is implicit.

#7 --- THE SOLUTION $x^2 - 1 = y - \ln(y+1)$ IS IMPLICIT BECAUSE THE DEP. VAR. y IS NOT GIVEN AS A FUNCTION OF ONLY x. INSTEAD, THE x & y ARE "ALL MIXED UP" IN THE EQUATION.

9. (10 points) A portion of the slope field for a differential equation is shown below.



(a) Estimate the slope of the solution curve passing through $(1,1)$.

THE LINE SEGMENT NEAREST TO $(1,1)$ IS PRACTICALLY HORIZONTAL. THE SOLUTION CURVE THROUGH $(1,1)$ HAS SLOPE ≈ 0.

(b) Consider the solution that satisfies $y(-3) = 2$. Would you expect that solution to have relative maximum and minimum values? Explain.

YES, SEE THE CURVE ABOVE WITH LABEL (b).

(c) Consider the solution that satisfies $y(0) = -4$. Would you expect that solution to have relative maximum and minimum values? Explain.

NO, SEE THE CURVE ABOVE WITH LABEL (c)

(d) Based on the available information, what would you expect any solution to do as x approaches ∞? Explain.

$$\lim_{x \to \infty} y(x) = \infty$$

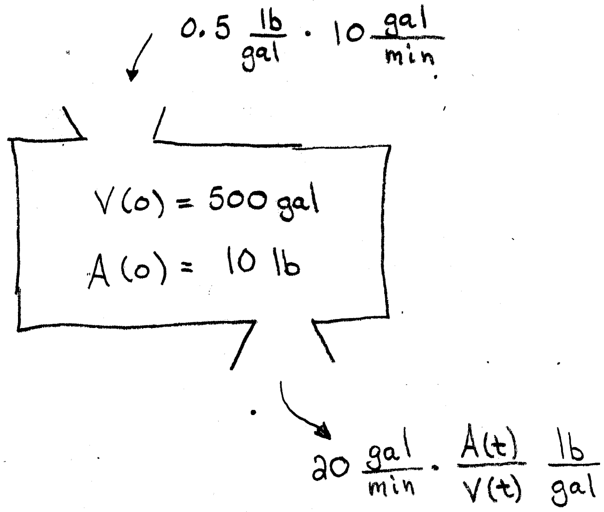
(e) Give two reasons to argue that the given slope field cannot be for the equation $dy/dx = -y/x$.

① FOR $\frac{dy}{dx} = -\frac{y}{x}$, ALL SLOPES WOULD BE NEGATIVE IN QUAD I. THIS IS NOT WHAT IS HAPPENING!

② FOR $\frac{dy}{dx} = -\frac{y}{x}$, SOLUTION CURVES WOULD HAVE VERTICAL TANGENTS ALONG $X=0$. NOT HAPPENING HERE!

The following problems are take-home problems. They are due February 18. You must work on your own.

10. (8 points) A tank is initially filled with 500 gal of a salt solution that contains 10 lb of salt. A salt solution containing 0.5 lb of salt per gallon enters the tank at 10 gal/min and is uniformly mixed. The mixed solution leaves the tank at 20 gal/min. Let $A(t)$ denote the amount of salt in the tank after t minutes. Set up and solve the appropriate initial value problem to determine $A(t)$. Find the maximum amount of salt in the tank.



$A(t)$ = AMOUNT OF SALT AT TIME t

$V(t)$ = VOLUME AT TIME t

$= 500 - 10t, \quad 0 \le t \le 50$

$$\frac{dA}{dt} = 5 - \frac{20A}{500-10t}, \quad A(0) = 10$$

$$\frac{dA}{dt} + \frac{2}{50-t} A = 5$$

$$\mu(t) = e^{\int \frac{2}{50-t} dt} = e^{-2 \ln|50-t|} = (50-t)^{-2}$$

$$(50-t)^{-2} A(t) = \int 5(50-t)^{-2} dt = 5(50-t)^{-1} + C$$

$A(0) = 10$

$$\Rightarrow \frac{10}{50^2} - \frac{5}{50} = C$$

$$C = \frac{-240}{2500}$$

$$\boxed{A(t) = 5(50-t) - \frac{24}{250}(50-t)^2 = -\frac{12}{125}t^2 + \frac{23}{5}t + 10}$$

$$A'(t) = -\frac{24}{125}t + \frac{23}{5}$$

$$A'(t) = 0 \Rightarrow t = \frac{23 \cdot 25}{24} = \frac{575}{24} = 23.958\overline{3} \text{ min}$$

$$\boxed{A\left(\frac{575}{24}\right) = \frac{3125}{48} \approx 65.104 \text{ lb}}$$

THIS IS A MAX ---

THE graph OF A IS A DOWNWARD OPENING PARABOLA.

11. (8 points) A chemical reaction is called a *second-order reaction* if the rate of change of the amount of the reactant, A, is proportional to the amount squared:

$$\frac{dA}{dt} = kA^2, \quad A(0) = A_0.$$

(a) Solve the initial value problem to find a formula for $A(t)$.

$$\frac{1}{A^2}\,dA = k\,dt$$

$$-\frac{1}{A} = kt + C$$

$$A(0) = A_0 \Rightarrow -\frac{1}{A_0} = C$$

$$-\frac{1}{A} = kt - \frac{1}{A_0}$$

$$A = \frac{1}{\frac{1}{A_0} - kt}$$

$$\boxed{A(t) = \frac{A_0}{1 - A_0 kt}}$$

(b) You are observing a second-order reaction involving the chemical butadiene. You begin your experiment (at $t = 0$) with 0.0100 kg of butadiene and after 10 seconds, there are 0.00625 kg of butadiene remaining. Find the constant k and write the complete formula for $A(t)$.

$$A_0 = 0.01$$

$$A(10) = 0.00625 = \frac{0.01}{1 - 0.01k(10)}$$

$$1 - 0.1k = 1.6$$

$$k = -6$$

$$\boxed{A(t) = \frac{0.01}{1 + 0.06t}}$$

(c) How long after you began the experiment will 0.00500 kg of butadiene remain?

$$\frac{0.01}{1 + 0.06t} = 0.005 \Rightarrow 1 + 0.06t = 2$$

$$t = \frac{1}{0.06} = 16.\overline{6} \text{ sec}$$

$$\approx \boxed{16.7 \text{ sec}}$$