MTH 240 - Assignment 7 Key

#1 $(x-2)y' + y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$0 = (x-2)y' + y = \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} 2 n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$$

SAME TO START AT $n=0$

REPLACE $n-1$ WITH n

$$= \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+1)a_n - 2(n+1)a_{n+1}\right] x^n$$

$$\Rightarrow \quad (n+1)a_n - 2(n+1)a_{n+1} = 0 \quad \text{FOR } n = 0, 1, 2, \dots$$

$$\Rightarrow \quad a_{n+1} = \frac{1}{2} a_n \quad \text{FOR } n = 0, 1, 2, 3, \dots$$

$\boxed{a_0 \text{ IS ARBITRARY}}$

$a_1 = \frac{1}{2} a_0$

$a_2 = \frac{1}{4} a_0$

$a_3 = \frac{1}{8} a_0$

$\vdots$

$$\boxed{a_n = \frac{1}{2^n} a_0, \quad n = 1, 2, 3, \dots}$$

$$y(x) = a_0\left(1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{16}x^4 + \frac{1}{32}x^5 + \frac{1}{64}x^6 + \frac{1}{128}x^7 + \frac{1}{256}x^8 + \cdots\right) = a_0 \sum_{n=0}^{\infty} \frac{1}{2^n} x^n$$

#2 $(x-10)y' + y = 0$

THIS PROBLEM IS IDENTICAL TO #1 EXCEPT THE "2" IS REPLACED BY A "10".

$$y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow \boxed{\begin{array}{l} a_0 = \text{ARBITRARY} \\ a_n = \frac{1}{10^n} a_0, \quad n = 1, 2, 3, \dots \end{array}}$$

$$\Rightarrow y(x) = a_0 \sum_{n=0}^{\infty} \left(\frac{x}{10}\right)^n = \frac{a_0}{1 - \frac{x}{10}} = \frac{C}{10 - x}, \quad |x| < 10$$

#3 $(x^2-3)y'' + 2xy' = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$0 = x^2 y'' - 3y'' + 2xy'$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^n - \underbrace{\sum_{n=2}^{\infty} 3n(n-1) a_n x^{n-2}}_{\text{REPLACE } n-2 \text{ WITH } n} + \sum_{n=1}^{\infty} 2n a_n x^n$$

$$= \underbrace{\sum_{n=2}^{\infty} n(n-1) a_n x^n}_{\text{SAME TO START AT } n=0} - \sum_{n=0}^{\infty} 3(n+2)(n+1) a_{n+2} x^n + \underbrace{\sum_{n=1}^{\infty} 2n a_n x^n}_{\text{SAME TO START AT } n=0}$$

#3 CONTINUED...

$$= \sum_{n=0}^{\infty} \left[n(n-1)\, a_n - 3(n+2)(n+1)\, a_{n+2} + 2n\, a_n \right] x^n$$

$$\Rightarrow \quad n(n+1)\, a_n - 3(n+2)(n+1)\, a_{n+2} = 0 \; ; \quad n = 0, 1, 2, 3, \ldots$$

$$\Rightarrow \quad a_{n+2} = \frac{n}{3(n+2)}\, a_n \; ; \quad n = 0, 1, 2, 3, \ldots$$

a_0 AND a_1 ARE ARBITRARY.

MINIMUM RADIUS OF CONVERGENCE ?

$$y'' + \frac{2x}{x^2-3}\, y' = 0$$

$x^2 - 3 = 0 \Rightarrow x = \pm\sqrt{3}$ SINGULAR PTS!

DISTANCE FROM CENTER $= \sqrt{3}$

RADIUS OF CONVERGENCE IS AT LEAST $\sqrt{3}$.

#4 $$5y'' - 2xy' + 10y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$0 = 5y'' - 2xy' + 10y$$

$$= \sum_{n=2}^{\infty} 5n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} 10a_n x^n$$

REPLACE $n-2$ WITH n

SAME TO START WITH $n=0$

$$= \sum_{n=0}^{\infty} 5(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} 10a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[5(n+2)(n+1)a_{n+2} - 2(n-5)a_n \right] x^n$$

$$\Rightarrow \quad 5(n+2)(n+1)a_{n+2} - 2(n-5)a_n = 0; \quad n = 0,1,2,3,\ldots$$

$$\Rightarrow \quad a_{n+2} = \frac{2(n-5)}{5(n+2)(n+1)} a_n; \quad n = 0,1,2,3,\ldots$$

a_0 & a_1 ARE ARBITRARY

MIN. RADIUS OF CONVERGENCE ?

$$y'' - \frac{2x}{5}y' + \frac{10}{5}y = 0$$

COEFFICIENTS ARE ANALYTIC EVERYWHERE.

No SINGULAR PTS.

RADIUS OF CONVERGENCE IS ∞.

#5 $y'' - x^2 y' - 3xy = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$0 = y'' - x^2 y' - 3xy$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \underbrace{\sum_{n=1}^{\infty} n a_n x^{n+1} - \sum_{n=0}^{\infty} 3 a_n x^{n+1}}_{\text{REPLACE } n+1 \text{ WITH } n}$$

REPLACE $n-2$ WITH n

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n - \sum_{n=1}^{\infty} 3 a_{n-1} x^n$$

$$= 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n - 3a_0 x - \sum_{n=2}^{\infty} 3 a_{n-1} x^n$$

$$= 2a_2 + (6a_3 - 3a_0)x + \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} - (n-1) a_{n-1} - 3 a_{n-1} \right] x^n$$

$$\Rightarrow a_2 = 0, \quad 6a_3 = 3a_0, \quad (n+2)(n+1) a_{n+2} - (n+2) a_{n-1} = 0; \quad n = 2, 3, 4, \ldots$$

$$\Rightarrow \boxed{a_2 = 0, \quad a_3 = \frac{1}{2} a_0, \quad a_{n+2} = \frac{1}{n+1} a_{n-1}; \quad n = 2, 3, 4, \ldots \qquad a_0 \text{ \& } a_1 \text{ ARE ARBITRARY.}}$$

MIN. RADIUS OF CONVERGENCE ?

THERE ARE NO SINGULAR PTS: $\boxed{\text{RADIUS OF CONVERGENCE IS } \infty.}$

#6 $x^3 y' = 2y$

a) Assume $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$$0 = x^3 y' - 2y$$

$$= \sum_{n=1}^{\infty} n a_n x^{n+2} - \sum_{n=0}^{\infty} 2 a_n x^n$$

Replace $n+2$ with n

$$= \sum_{n=3}^{\infty} (n-2) a_{n-2} x^n - \sum_{n=0}^{\infty} 2 a_n x^n$$

$$= \sum_{n=3}^{\infty} (n-2) a_{n-2} x^n - 2a_0 - 2a_1 x - 2a_2 x^2 - \sum_{n=3}^{\infty} 2 a_n x^n$$

$\Longrightarrow$ $a_0 = a_1 = a_2 = 0$, $(n-2) a_{n-2} - 2a_n = 0$; $n = 3, 4, 5, \ldots$

$$a_n = \frac{n-2}{2} a_{n-2}; \quad n = 3, 4, 5, \ldots$$

All a_n's are zero!

We found $y(x) = 0$. This is a particular solution, but not the general solution.

b) $y' - \frac{2}{x^3} y = 0$

$$\mu(x) = e^{\int -\frac{2}{x^3}\,dx} = e^{1/x^2}$$

$$e^{1/x^2} y(x) = \int 0\,dx = C$$

$$\boxed{y(x) = \frac{C}{e^{1/x^2}}}$$

c) WE SHOULD NOT HAVE EXPECTED A POWER SERIES CENTERED AT $X=0$ TO PROVIDE A SOLUTION BECAUSE $X=0$ IS A SINGULAR POINT.

#7

a) $y'' + \frac{7x}{x+4} y' - \frac{5(x+2)}{x+4} y = 0$

$X = -4$ IS THE ONLY SINGULAR PT.

DISTANCE FROM $X=0$ TO $X=-4$ IS 4 UNITS.

RADIUS OF CONVERGENCE IS AT LEAST 4.

b) $y'' + \frac{8}{x^2+1} y' + \frac{x^2-1}{x^2+1} y = 0$

SINGULAR PTS ARE $X = \pm i$.

DISTANCE FROM $X=0$ TO $X=\pm i$ IS 1 UNIT.

RADIUS OF CONVERGENCE IS AT LEAST 1.