MTH 240 Assignment 10 key

1) $y'' - 2y' + 5y = 0; \quad y(0) = 2, \; y'(0) = 4$

Let $\mathcal{L}\{y\} = Y$.

$$s^2 Y(s) - 2s - 4 - 2(sY(s) - 2) + 5Y(s) = 0$$

$$(s^2 - 2s + 5)\,Y(s) - 2s = 0$$

$$Y(s) = \frac{2s}{s^2 - 2s + 5} = \frac{2s}{(s-1)^2 + 4} = \frac{2(s-1)}{(s-1)^2+4} + \frac{2}{(s-1)^2+4}$$

$$\boxed{y(t) = 2e^t \cos 2t + e^t \sin 2t}$$

2) $y'' - 4y = 0; \quad y(0) = 0, \; y'(0) = 5$

Let $\mathcal{L}\{y\} = Y$.

$$s^2 Y(s) - 5 - 4Y(s) = 0$$

$$(s^2 - 4)\,Y(s) = 5$$

$$Y(s) = \frac{5}{s^2 - 4} = \frac{A}{s-2} + \frac{B}{s+2}$$

Cover up...
$A = \frac{5}{4}, \; B = -\frac{5}{4}$

$$Y(s) = \frac{5}{4}\,\frac{1}{s-2} - \frac{5}{4}\,\frac{1}{s+2}$$

$$\boxed{y(t) = \frac{5}{4} e^{2t} - \frac{5}{4} e^{-2t} = \frac{5}{2} \sinh t}$$

3) $x''' + x'' - 6x' = 0; \quad x(0) = 0, \; x'(0) = 1, \; x''(0) = 1$

Let $\mathcal{L}\{x\} = X$

$$\left(s^3 X(s) - s - 1\right) + \left(s^2 X(s) - 1\right) - 6\left(s X(s)\right) = 0$$

$$\left(s^3 + s^2 - 6s\right) X(s) = s + 2$$

$$X(s) = \frac{s+2}{s(s+3)(s-2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-2}$$

Cover up... $A = -\frac{1}{3}$

$B = -\frac{1}{15}$

$C = \frac{2}{5}$

$$X(s) = \frac{-1/3}{s} - \frac{1/15}{s+3} + \frac{2/5}{s-2}$$

$$\boxed{x(t) = -\frac{1}{3} - \frac{1}{15} e^{-3t} + \frac{2}{5} e^{2t}}$$

4) $x' = -x + y$, $x(0) = 0$

$y' = 2x$, $y(0) = 1$

LET $\mathcal{L}\{x\} = X$ AND $\mathcal{L}\{y\} = Y$.

$$sX(s) = -X(s) + Y(s) \longrightarrow Y(s) = (s+1)X(s)$$

$$sY(s) - 1 = 2X(s)$$

$$s(s+1)X(s) - 1 = 2X(s)$$

$$(s^2 + s - 2)X(s) = 1$$

$$X(s) = \frac{1}{(s+2)(s-1)} = \frac{-1/3}{s+2} + \frac{1/3}{s-1}$$

RESUB ...

$$Y(s) = \frac{s+1}{(s+2)(s-1)} = \frac{1/3}{s+2} + \frac{2/3}{s-1}$$

$$\boxed{\begin{aligned} x(t) &= -\tfrac{1}{3}e^{-2t} + \tfrac{1}{3}e^{t} \\ y(t) &= \tfrac{1}{3}e^{-2t} + \tfrac{2}{3}e^{t} \end{aligned}}$$

5) $$t * t^2 = \int_0^t v^2 (t-v)\, dv$$

$$= \int_0^t (v^2 t - v^3)\, dv = \frac{1}{3} v^3 t - \frac{1}{4} v^4 \Big|_{v=0}^{v=t}$$

$$= \frac{1}{3} t^4 - \frac{1}{4} t^4 = \boxed{\frac{1}{12} t^4}$$

6) $$F(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} \cdot \frac{1}{s+2}$$

$$f(t) = e^{-t} * e^{-2t} = \int_0^t e^{-2v} e^{-(t-v)}\, dv$$

$$= \int_0^t e^{-t} e^{-v}\, dv = e^{-t} \int_0^t e^{-v}\, dv = e^{-t} \left(-e^{-v} \Big|_0^t \right)$$

$$= \boxed{e^{-t} - e^{-2t}}$$

7) $y'' - 2y' + y = g(t)\ ;\quad y(0) = -1,\ y'(0) = 1$

LET $\mathcal{L}\{y\} = Y$ AND $\mathcal{L}\{g\} = G$

$$\left(s^2 Y(s) + s - 1\right) - 2\left(s\,Y(s) + 1\right) + Y(s) = G(s)$$

$$\left(s^2 - 2s + 1\right) Y(s) + s - 3 = G(s)$$

$$Y(s) = \frac{G(s)}{(s-1)^2} - \frac{s-3}{(s-1)^2}$$

$$= \frac{1}{(s-1)^2}\,G(s) - \left[\frac{s-1}{(s-1)^2} - \frac{2}{(s-1)^2}\right]$$

$$= \frac{1}{(s-1)^2}\,G(s) - \frac{1}{s-1} + \frac{2}{(s-1)^2}$$

$$y(t) = t * g(t) - e^t + 2te^t$$

$$\boxed{y(t) = \int_0^t g(v)\,(t-v)\,dv - e^t + 2te^t}$$