1) $(x-3)y' + y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$0 = xy' - 3y' + y$$

$$= \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} 3n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$$

Replace n with n+1

$$= \sum_{n=1}^{\infty} n a x^n - \sum_{n=0}^{\infty} 3(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 3(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+1) a_n - 3(n+1) a_{n+1} \right] x^n$$

$$(n+1) a_n = 3(n+1) a_{n+1} \qquad n = 0, 1, 2, \cdots$$

$$a_{n+1} = \frac{1}{3} a_n; \quad n = 0, 1, 2, \cdots$$

a_0 ARBITRARY

$$a_1 = \frac{1}{3} a_0$$
$$a_2 = \frac{1}{3^2} a_0$$
$$a_3 = \frac{1}{3^3} a_0$$
$$\vdots$$

$$a_n = \frac{1}{3^n} a_0$$

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{1}{3^n} x^n$$

$$y(x) = a_0 \left(1 + \frac{1}{3} x + \frac{1}{3^2} x^2 + \frac{1}{3^3} x^3 + \cdots + \frac{1}{3^8} x^8 + \cdots \right)$$

2) $(x+5)y' + y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$0 = xy' + 5y' + y$$

$$= \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} 5n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$$

REPLACE n WITH $n+1$

$$= \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 5(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 5(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[n a_n + 5(n+1) a_{n+1} + a_n \right] x^n$$

$$n a_n + 5(n+1) a_{n+1} + a_n = 0; \quad n = 0, 1, 2, \ldots$$

$$a_{n+1} = -\frac{1}{5} a_n; \quad n = 0, 1, 2, \ldots$$

a_0 ARBITRARY

$a_1 = -\frac{1}{5} a_0$

$a_2 = \frac{1}{5^2} a_0$

$a_3 = -\frac{1}{5^3} a_0$

$a_4 = \frac{1}{5^4} a_0$

$a_5 = -\frac{1}{5^5} a_0$

$a_6 = \frac{1}{5^6} a_0$

$\vdots$

$a_n = \frac{(-1)^n}{5^n} a_0$

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n} x^n$$

A

$$y(x) = a_0 \left(1 - \frac{1}{5}x + \frac{1}{5^2}x^2 - \frac{1}{5^3}x^3 + \frac{1}{5^4}x^4 - \frac{1}{5^5}x^5 + \cdots \right)$$

3) $(x^2-3)y'' + 2xy' = 0$

$$y'' + \frac{2x}{x^2-3}y' = 0 \qquad x^2-3=0$$

$$x = \pm\sqrt{3}$$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

DISTANCE FROM CENTER $x=0$ TO NEAREST SING. PT $x = \pm\sqrt{3}$ IS $\sqrt{3}$ UNITS.

$$R \geq \sqrt{3}$$

$$0 = (x^2-3)y'' + 2xy'$$

$$= x^2y'' - 3y'' + 2xy'$$

$$= \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=2}^{\infty} 3n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} 2na_n x^n$$

$n \leftrightarrow n+2$

$$= \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} 3(n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} 2na_n x^n$$

$$= \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} 3(n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} 2na_n x^n$$

$$= \sum_{n=0}^{\infty} \left[n(n-1)a_n - 3(n+2)(n+1)a_{n+2} + 2na_n \right] x^n$$

$$(n^2+n)a_n - 3(n+2)(n+1)a_{n+2} = 0; \quad n = 0, 1, 2, 3, \ldots$$

$$a_{n+2} = \frac{n(n+1)}{3(n+2)(n+1)} a_n; \quad n = 0, 1, 2, \ldots$$

$$a_{n+2} = \frac{n}{3(n+2)} a_n; \quad n = 0, 1, 2, \ldots$$

a_0 & a_1 ARBITRARY

4) $y'' + 3xy' - y = 0; \quad y(0) = 2, \; y'(0) = 0$

$$y := \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$0 = y'' + 3xy' - y$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} 3n a_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

$n \leftrightarrow n+2$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} 3n a_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 3n a_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + 3n a_n - a_n \right] x^n$$

$$(n+2)(n+1) a_{n+2} + 3n a_n - a_n = 0; \quad n = 0, 1, 2, \ldots$$

$$a_{n+2} = \frac{(1-3n)}{(n+2)(n+1)} a_n; \quad n = 0, 1, 2, 3, \ldots$$

a_0, a_1 ARBITRARY

a_0 ARB

a_1 ARB

$a_2 = \frac{1}{2} a_0$

$a_3 = \frac{-2}{6} a_1$

$a_4 = \frac{-5}{12} \cdot \frac{1}{2} a_0$

$a_5 = \frac{-8}{20} \cdot \frac{-2}{6} a_1$

$a_6 = \frac{-11}{30} \cdot \frac{-5}{12} \cdot \frac{1}{2} a_0$

$$y(x) = a_0 + \frac{1}{2} a_0 x^2 - \frac{5}{24} a_0 x^4 + \frac{11}{144} a_0 x^6 - \cdots$$

$$+ a_1 x - \frac{1}{3} a_1 x^3 + \frac{2}{15} a_1 x^5 - \cdots$$

$y(0) = 2 \Rightarrow a_0 = 2$

$y'(0) = 0 \Rightarrow a_1 = 0$

$$y(x) = 2 + x^2 - \frac{5}{12} x^4 + \frac{11}{72} x^6 - \cdots$$

5) $5y'' - 2xy' + 10y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$0 = 5y'' - 2xy' + 10y$$

$$= \sum_{n=2}^{\infty} 5n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 10 a_n x^n$$

Replace $n-2$ with n

Same to start with $n=0$

$$= \sum_{n=0}^{\infty} 5(n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 10 a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[5(n+2)(n+1) a_{n+2} - 2(n-5) a_n \right] x^n$$

$$\Rightarrow \quad 5(n+2)(n+1) a_{n+2} - 2(n-5) a_n = 0 ; \quad n = 0, 1, 2, 3, \ldots$$

$$\Rightarrow \quad a_{n+2} = \frac{2(n-5)}{5(n+2)(n+1)} a_n ; \quad n = 0, 1, 2, 3, \ldots$$

a_0 & a_1 ARE ARBITRARY

MIN. RADIUS OF CONVERGENCE ?

$$y'' - \frac{2x}{5} y' + \frac{10}{5} y = 0$$

COEFFICIENTS ARE ANALYTIC EVERYWHERE.

No SINGULAR PTS.

RADIUS OF CONVERGENCE IS ∞.

6) $x^3 y' = 2y$

a) Assume $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$$0 = x^3 y' - 2y$$

$$= \sum_{n=1}^{\infty} n a_n x^{n+2} - \sum_{n=0}^{\infty} 2 a_n x^n$$

Replace $n+2$ with n

$$= \sum_{n=3}^{\infty} (n-2) a_{n-2} x^n - \sum_{n=0}^{\infty} 2 a_n x^n$$

$$= \sum_{n=3}^{\infty} (n-2) a_{n-2} x^n - 2a_0 - 2a_1 x - 2a_2 x^2 - \sum_{n=3}^{\infty} 2 a_n x^n$$

$$\Rightarrow \quad a_0 = a_1 = a_2 = 0, \quad (n-2) a_{n-2} - 2 a_n = 0; \quad n = 3, 4, 5, \ldots$$

$$a_n = \frac{n-2}{2} a_{n-2}; \quad n = 3, 4, 5, \ldots$$

↑ All a_n's are zero!

We found $y(x) = 0$. This is a particular solution, but not the general solution.

b) $y' - \frac{2}{x^3} y = 0$

$$\mu(x) = e^{\int -\frac{2}{x^3}\,dx} = e^{1/x^2}$$

$$e^{1/x^2} y(x) = \int 0\,dx = C$$

$$\boxed{y(x) = \frac{C}{e^{1/x^2}}}$$

c) WE SHOULD NOT HAVE EXPECTED A POWER SERIES CENTERED AT $x=0$ TO PROVIDE A SOLUTION BECAUSE $x=0$ IS A SINGULAR POINT.

7) a) $y'' + \frac{7x}{x+4} y' - \frac{5(x+2)}{x+4} y = 0$

$x=-4$ IS THE ONLY SINGULAR PT.

DISTANCE FROM $x=0$ TO $x=-4$ IS 4 UNITS.

RADIUS OF CONVERGENCE IS AT LEAST 4.

b) $y'' + \frac{8}{x^2+1} y' + \frac{x^2-1}{x^2+1} y = 0$

SINGULAR PTS ARE $x = \pm i$.

DISTANCE FROM $x=0$ TO $x=\pm i$ IS 1 UNIT.

RADIUS OF CONVERGENCE IS AT LEAST 1.