MTH 240 Assignment #3 Key

1.) $y' = 2y + x(e^{3x} - e^{2x})$, $\quad y(0) = 2$

$$y' - 2y = x(e^{3x} - e^{2x})$$

$$\mu(x) = e^{\int -2\,dx} = e^{-2x} \Rightarrow e^{-2x} y = \int e^{-2x} x (e^{3x} - e^{2x})\,dx$$

$$= \int (xe^x - x)\,dx = \underbrace{\int xe^x\,dx} - \frac{x^2}{2} + C$$

$u = x \qquad du = dx$

$dv = e^x\,dx \qquad v = e^x$

$$= xe^x - e^x - \frac{x^2}{2} + C$$

$$y(x) = xe^{3x} - e^{3x} - \frac{1}{2}x^2 e^{2x} + Ce^{2x}$$

$$y(0) = 2 \Rightarrow -1 + C = 2 \Rightarrow C = 3$$

$$\boxed{y(x) = (x-1)e^{3x} + \left(3 - \frac{1}{2}x^2\right) e^{2x}}$$

2.) $x \dfrac{dy}{dx} + 4y = x^3 - x \Rightarrow \dfrac{dy}{dx} + \dfrac{4}{x} y = x^2 - 1$, $\quad x \neq 0$

$$\mu(x) = e^{\int \frac{4}{x} dx} = e^{4 \ln|x|} = x^4$$

$$x^4 \cdot y = \int (x^6 - x^4)\,dx = \frac{1}{7}x^7 - \frac{1}{5}x^5 + C$$

$$\boxed{y(x) = \frac{1}{7}x^3 - \frac{1}{5}x + Cx^{-4}}$$

3.) $\cos^2 x \dfrac{dy}{dx} + y = 1, \quad y(0) = -3$

$$\frac{dy}{dx} + \sec^2 x \; y = \sec^2 x$$

$$\mu(x) = e^{\int \sec^2 x \, dx} = e^{\tan x} \Rightarrow e^{\tan x} y = \int \sec^2 x \, e^{\tan x} dx$$

$u = \tan x$
$du = \sec^2 x \, dx$

$$= e^{\tan x} + C$$

$$y(x) = 1 + Ce^{-\tan x}$$

$y(0) = -3 \Rightarrow 1 + C = -3 \Rightarrow C = -4$

$$\boxed{y(x) = 1 - 4e^{-\tan x}}$$

4.) $x^2 y' + x(x+2)y = e^x$

$$y' + \frac{x+2}{x} y = \frac{e^x}{x^2}, \quad x \neq 0$$

$$\mu(x) = e^{\int 1 + \frac{2}{x} dx} = e^{x + 2\ln|x|} = x^2 e^x$$

$$x^2 e^x y = \int e^{2x} dx = \frac{1}{2} e^{2x} + C$$

$$\boxed{y(x) = \frac{e^x}{2x^2} + \frac{C}{x^2 e^x}}$$

5.)

$$\frac{1}{2}\,\frac{\text{lb}}{\text{gal}} \cdot 6\,\frac{\text{gal}}{\text{min}} = 3\,\frac{\text{lb}}{\text{min}}$$

$V(0) = 100\text{ gal}$

$A(0) = 10\text{ lb}$

$V(t) = 100 + 2t$ = VOLUME AT t

$A(t)$ = AMOUNT OF SALT AT t

$$4\,\frac{\text{gal}}{\text{min}} \cdot \frac{A(t)}{V(t)}\,\frac{\text{lb}}{\text{gal}} = \frac{4A}{100+2t}$$

$$\frac{dA}{dt} = 3 - \frac{2A}{50+t}, \quad A(0) = 10$$

$$\frac{dA}{dt} + \frac{2}{50+t}A = 3$$

$$\mu(t) = e^{\int \frac{2}{50+t}dt} = e^{2\ln|50+t|} = (50+t)^2$$

$$(50+t)^2 A = \int 3(50+t)^2\,dt$$

$$= (50+t)^3 + C$$

$$A(t) = 50 + t + \frac{C}{(50+t)^2}$$

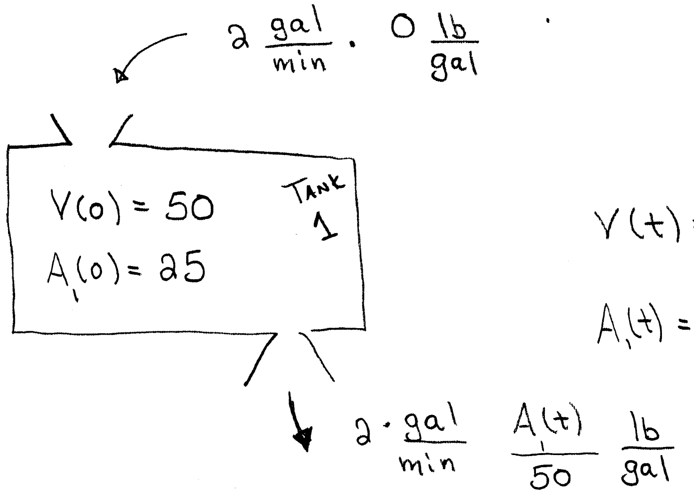
$$A(0) = 10 \Rightarrow 10 = 50 + \frac{C}{2500} \Rightarrow C = -40(2500)$$

$$= -100{,}000$$

$$\boxed{A(t) = 50 + t - \frac{100{,}000}{(50+t)^2}}$$

$$\boxed{A(30) = 80 - \frac{100{,}000}{6400} = 64.375\text{ lb}}$$

6)

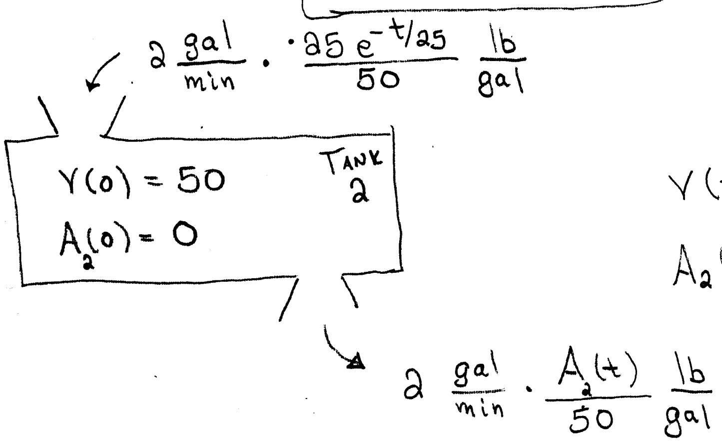


$V(t) = 50$ (CONST)

$A_1(t)$ = AMOUNT OF SALT IN TANK 1 AT t

$$\underbrace{\frac{dA_1}{dt} = -\frac{A_1}{25}, \quad A_1(0) = 25}_{\text{EXPONENTIAL DECAY!}}$$

$$\boxed{A_1(t) = 25e^{-t/25}}$$



$V(t) = 50$ (CONST)

$A_2(t)$ = AMOUNT OF SALT IN TANK 2 AT t

$$\frac{dA_2}{dt} = e^{-t/25} - \frac{A_2}{25}, \quad A_2(0) = 0$$

$$\frac{dA_2}{dt} + \frac{1}{25}A_2 = e^{-t/25}$$

$$\mu(t) = e^{\int \frac{1}{25}dt} = e^{t/25}$$

$$e^{t/25}A_2(t) = \int 1\,dt = t + C$$

$$A_2(t) = (t + C)e^{-t/25}$$

$$A_2(0) = 0 \Rightarrow C = 0$$

$$\boxed{A_2(t) = te^{-t/25}}$$

$$A_2'(t) = e^{-t/25} - \frac{1}{25}te^{-t/25}$$

$$= 0 \Rightarrow \boxed{t = 25 \text{ min}}$$

Apply 1ST OR 2ND DERIVATIVE TESTS TO CHECK THAT THIS INDEED GIVES A MAX.

7.) $\underbrace{(4y+2x-5)}_{M}\,dx + \underbrace{(6y+4x-1)}_{N}\,dy = 0, \quad y(-1)=2$

$$\frac{\partial M}{\partial y} = 4 = \frac{\partial N}{\partial x} \Rightarrow \text{EQUATION IS EXACT}$$

$F_x(x,y) = 4y+2x-5 \Rightarrow F(x,y) = 4xy + x^2 - 5x + g(y)$

$F_y(x,y) = 6y+4x-1 \Rightarrow F(x,y) = 3y^2 + 4xy - y + h(x)$

$F(x,y) = 4xy + x^2 - 5x + 3y^2 - y$

SOLUTION:

$4xy + x^2 - 5x + 3y^2 - y = C$

$y(-1) = 2 \Rightarrow 4(-1)(2) + (-1)^2 - 5(-1) + 3(2)^2 - 2 = C$

$C = 8$

$$\boxed{4xy + x^2 - 5x + 3y^2 - y = 8}$$

8.) $\underbrace{\left(ye^{xy} - \frac{1}{y}\right)}_{M}dx + \underbrace{\left(xe^{xy} + \frac{x}{y^2}\right)}_{N}dy = 0$

$$\frac{\partial M}{\partial y} = e^{xy} + xye^{xy} + \frac{1}{y^2} = \frac{\partial N}{\partial x} \Rightarrow \text{EQUATION IS EXACT.}$$

$F_x(x,y) = ye^{xy} - \frac{1}{y} \Rightarrow F(x,y) = e^{xy} - \frac{x}{y} + g(y)$

$F_y(x,y) = xe^{xy} + \frac{x}{y^2} \Rightarrow F(x,y) = e^{xy} - \frac{x}{y} + h(x)$

$F(x,y) = e^{xy} - \frac{x}{y}$

SOLUTION:

$e^{xy} - \frac{x}{y} = C$

9.) $\underbrace{(xy + y^2 + y)}_{M}\, dx + \underbrace{(x + 2y)}_{N}\, dy = 0$

$\frac{\partial M}{\partial y} = x + 2y + 1 \neq \frac{\partial N}{\partial x} = 1 \Rightarrow$ EQUATION IS <u>NOT</u> EXACT.

$\underbrace{(xye^x + y^2e^x + ye^x)}_{M}\, dx + \underbrace{(xe^x + 2ye^x)}_{N}\, dy = 0$

$\frac{\partial M}{\partial y} = xe^x + 2ye^x + e^x = \frac{\partial N}{\partial x} \Rightarrow$ NEW EQUATION <u>IS</u> EXACT.

$F_x(x,y) = xye^x + y^2e^x + ye^x \Rightarrow F(x,y) = y\underbrace{\int xe^x\,dx}_{} + y^2e^x + ye^x + g(y)$

$u = x \qquad du = dx$
$dv = e^x\,dx \quad v = e^x$

$= xye^x - ye^x + y^2e^x + ye^x + g(y)$

$F_y(x,y) = xe^x + 2ye^x \Rightarrow F(x,y) = xye^x + y^2e^x + h(x)$

$F(x,y) = xye^x + y^2e^x$

$\boxed{\text{SOLUTION: } xye^x + y^2e^x = C}$