MTH 240 Assignment #11 key

1) $f(t) = t, \; g(t) = t^3$

$$f(t) * g(t) = g(t) * f(t) = \int_0^t \tau^3 (t-\tau)\, d\tau$$

$$= t \int_0^t \tau^3 \, d\tau - \int_0^t \tau^4 \, d\tau$$

$$= \frac{1}{4} t^5 - \frac{1}{5} t^5 = \boxed{\frac{1}{20} t^5}$$

2) $$F(s) = \frac{1}{(s-1)(s+5)} = \frac{1}{s-1} \cdot \frac{1}{s+5}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) * \mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}(t)$$

$$= e^t * e^{-5t}$$

$$= e^{-5t} * e^t = \int_0^t e^{-5\tau} e^{(t-\tau)}\, d\tau$$

$$= \int_0^t e^{-5\tau} e^t e^{-\tau}\, d\tau = e^t \int_0^t e^{-6\tau}\, d\tau$$

$$= e^t \left(-\frac{1}{6} e^{-6\tau} \Big|_0^t \right) = e^t \left(-\frac{1}{6} e^{-6t} + \frac{1}{6} \right)$$

$$= \boxed{-\frac{1}{6} e^{-5t} + \frac{1}{6} e^t}$$

3.) $y'' + 5y' + 4y = g(t); \quad y(0) = 1, \; y'(0) = 2$

$$s^2 Y(s) - s\overset{1}{y(0)} - \overset{2}{y'(0)} + 5s\,Y(s) - 5\overset{1}{y(0)} + 4\,Y(s) = G(s)$$

$$\underbrace{(s^2 + 5s + 4)}_{(s+4)(s+1)} Y(s) - s - 7 = G(s)$$

$$Y(s) = \frac{G(s)}{(s+4)(s+1)} + \frac{s+7}{(s+4)(s+1)}$$

$$= \left(\frac{-1/3}{s+4} + \frac{1/3}{s+1}\right) \cdot G(s) + \frac{-1}{s+4} + \frac{2}{s+1}$$

$$y(t) = \left(-\tfrac{1}{3}e^{-4t} + \tfrac{1}{3}e^{-t}\right) * g(t) - e^{-4t} + 2e^{-t}$$

$$\boxed{y(t) = 2e^{-t} - e^{-4t} + \frac{1}{3}\int_0^t \left(e^{-\tau} - e^{-4\tau}\right) g(t-\tau)\, d\tau}$$

4.) $t x'' - t x' + x = 2;\quad x(0) = 2,\ x'(0) = -1$

$$-\frac{d}{ds}\left(s^2 X(s) - s x(0) - x'(0)\right) + \frac{d}{ds}\left(s X(s) - x(0)\right) + X(s) = \frac{2}{s}$$

$$-\frac{d}{ds}\left(s^2 X - 2s + 1\right) + \frac{d}{ds}\left(s X - 2\right) + X = \frac{2}{s}$$

$$-2s X - s^2 X' + 2 + X + s X' + X = \frac{2}{s}$$

$$-(s^2 - s) X' - (2s - 2) X = \frac{2}{s} - 2 = \frac{-2s + 2}{s} = \frac{-2(s-1)}{s}$$

$$X' + \frac{2}{s} X = \frac{2}{s^2}$$

$$\mu(s) = e^{\int \frac{2}{s} ds} = e^{2 \ln |s|} = s^2$$

$$s^2 X = \int 2\, ds = 2s + C$$

$$X(s) = \frac{2}{s} + \frac{C}{s^2}$$

$$x(t) = 2 + Ct$$

$$x'(0) = -1 \Rightarrow C = -1$$

$$\boxed{x(t) = 2 - t}$$

5.) $f(x) = x$ on $(-\pi, \pi)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\, dx = 0$$

ODD FUNCTION

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx\, dx = 0; \quad n = 1, 2, 3, \ldots$$

ODD FUNCTION

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\, dx$$

EVEN FUNCTION

$$= \frac{2}{\pi}\int_{0}^{\pi} x \sin nx\, dx$$

SIGNS	u & DERIVS	$\frac{dv}{dx}$ & ANTIS
+	x	$\sin nx$
−	1	$-\frac{1}{n}\cos nx$
+	0	$-\frac{1}{n^2}\sin nx$

$$= \frac{2}{\pi}\left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(-\frac{\pi}{n}\cos n\pi\right)$$

$$= -\frac{2}{n}(-1)^n$$

$$= \frac{2}{n}(-1)^{n+1}$$

$$b_n = \frac{2}{n}(-1)^{n+1}; \quad n = 1, 2, 3, \ldots$$

$$\boxed{f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx}$$