Math 240 - Test 1

September 12, 2024

Name ______________________

Score __________

Show all work to receive full credit. Supply explanations where necessary. Give explicit solutions when possible. All integration must be done by hand (showing work), unless otherwise specified.

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1. (10 points) State whether each equation is ordinary or partial, linear or nonlinear, and give its order. Also state which variable is the dependent variable.

   (a) $\dfrac{\partial U}{\partial t} = 2y\dfrac{\partial U}{\partial y} + 4x\dfrac{\partial^2 U}{\partial x^2}$

   (b) $6t^2\dfrac{d^2x}{dt^2} - 3t\dfrac{dx}{dt} + 4x = \ln(t)$

   (c) $\dfrac{dy}{dx} = \dfrac{y(2-3x)}{x(1-3y)}$

   (d) $y'' - 0.1(1-y^2)y' + 9y = 0$

2. (4 points) Newton's law of cooling states that the time rate of change of the temperature T of an object is proportional to the difference between its temperature and the surrounding (constant) temperature. Write a differential equation that models this situation.

3. (3 points) What is the difference between an explicit solution of a differential equation and an implicit solution?

4. (6 points) Consider the initial value problem (IVP) $\dfrac{dy}{dx} = 3y^{2/3}, \quad y(2) = 0.$

   (a) Verify that the constant function $y_1(x) \equiv 0$ is a solution of the IVP.

   (b) Verify that $y_2(x) = (x-2)^3$ is also a solution of the IVP.

   (c) Explain why the existence of multiple solutions does not contradict our existence and uniqueness theorem.

5. (12 points) Analyze each initial value problem to determine which one of these applies. **Do not attempt to solve the equations.**

   (A) A solution exists, but it is not guaranteed to be unique.

   (B) There is a unique solution.

   (C) A solution is not guaranteed to exist.

   Be sure to show work or explain.

   (a) $7xy' + y\sin x = \cos x^2, \quad y(1) = 3$

   (b) $y' + x\sqrt[3]{y} = 0, \quad y(\pi) = 0$

   (c) $(y-x)y' = y + x, \quad y(2) = 2$

6. (8 points) If a population P is growing *exponentially*, then the rate of change of P is proportional to P. Further assume that the population at $t = 0$ is P_0. Write the initial value problem corresponding to this situation and then solve it.

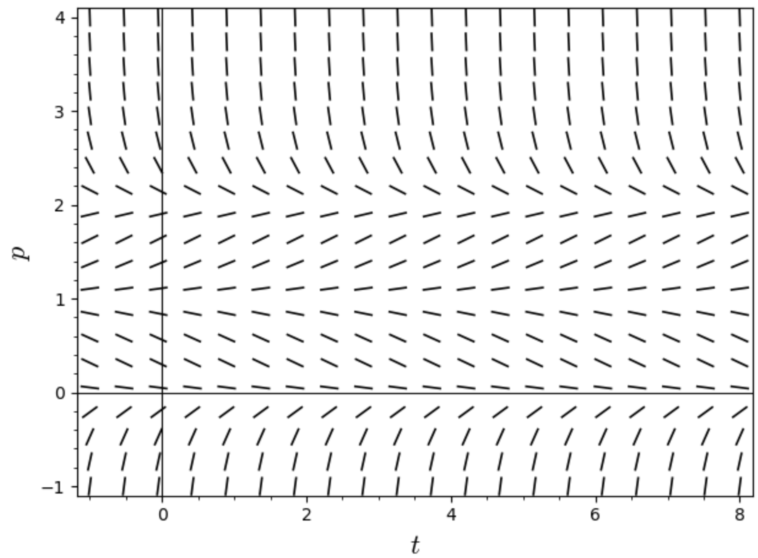
7. (10 points) Solve the initial value problem: $x^2 y' = y - xy, \quad y(-1) = -1$

8. (9 points) Find the value of the constant k so that the equation is exact. Then solve the equation with that choice of k.

$$(2xy^2 + ye^x + 7)\,dx + (2x^2y + ke^x - 1)\,dy = 0$$

9. (13 points) Solve the initial value problem: $\quad (x^2+9)\dfrac{dy}{dx} + xy = x, \quad y(0) = 4$

10. (10 points) Consider the differential equation $\frac{dp}{dt} = p(p-1)(2-p)$ for the population p (in thousands) of a certain species at time t. A portion of its slope field is shown below.



(a) What is the slope of the solution curve passing through $(t, p) = (0, 3)$?

(b) If the initial population is 3000 (that is, $p(0) = 3$), what can you say about the limiting population $\lim_{t\to\infty} p(t)$?

(c) If $p(0) = 1.5$, what is $\lim_{t\to\infty} p(t)$?

(d) If $p(0) = 0.5$, what is $\lim_{t\to\infty} p(t)$?

(e) Can a population of 900 ever increase to 1100?

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The following problems are due September 17. You must work on your own.

11. (7 points) Consider the initial value problem $y' - y = -x - 1, \quad y(0) = 1$.

    (a) Use Euler's method with $h = 0.1$ to estimate $y(0.3)$.

    (b) Use an Euler's method calculator (or other program) with $h = 0.001$ to approximate $y(0.3)$.

    (c) Find the exact solution of the IVP.

    (d) Use your exact solution to compute $y(0.3)$ and compare it to your approximation in parts (a) and (b).

12. (8 points) A tank initially contains 500 kg of salt dissolved in 5000 L of water. A salt solution containing 0.04 kg of salt per liter enters the tank at 20 L/min and is uniformly mixed. The mixed solution leaves the tank at 25 L/min. Let $A(t)$ denote the amount of salt in the tank after t minutes. Set up and solve the appropriate initial value problem to determine $A(t)$. Then find the amount of salt in the tank when the volume has decreased to 2000 L.