Math 240 - Test 1

September 12, 2024

Name Key

Score ______

Show all work to receive full credit. Supply explanations where necessary. Give explicit solutions when possible. All integration must be done by hand (showing work), unless otherwise specified.

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1. (10 points) State whether each equation is ordinary or partial, linear or nonlinear, and give its order. Also state which variable is the dependent variable.

   (a) $\dfrac{\partial U}{\partial t} = 2y\dfrac{\partial U}{\partial y} + 4x\dfrac{\partial^2 U}{\partial x^2}$

   PARTIAL, LINEAR, 2^{ND} ORDER.
   DEPENDENT VARIABLE IS U.

   (b) $6t^2\dfrac{d^2x}{dt^2} - 3t\dfrac{dx}{dt} + 4x = \ln(t)$

   ORDINARY, LINEAR, 2^{ND} ORDER.
   DEPENDENT VARIABLE IS X.

   (c) $\dfrac{dy}{dx} = \dfrac{y(2-3x)}{x(1-3y)}$

   ORDINARY, NONLINEAR, 1^{ST} ORDER.
   DEPENDENT VARIABLE IS y.

   (d) $y'' - 0.1(1-y^2)y' + 9y = 0$

   ORDINARY, NONLINEAR, 2^{ND} ORDER.
   DEPENDENT VARIABLE IS y.

2. (4 points) Newton's law of cooling states that the time rate of change of the temperature T of an object is proportional to the difference between its temperature and the surrounding (constant) temperature. Write a differential equation that models this situation.

   LET A = CONSTANT SURROUNDING TEMP.

   $$\boxed{\frac{dT}{dt} = k(T-A)},$$ WHERE k IS THE PROPORTIONALITY CONSTANT

3. (3 points) What is the difference between an explicit solution of a differential equation and an implicit solution?

AN EXPLICIT SOLUTION HAS THE DEPENDENT VARIABLE, SAY y, EXPLICITLY SOLVED FOR, $y = f(x)$. FOR EXAMPLE, $y = 3x + x^2 + e^x$

AN IMPLICIT SOLUTION HAS y DESCRIBED IN AN EQUATION IN WHICH IT IS NOT EXPLICITLY SOLVED FOR. FOR EXAMPLE, $xy + 3xy^2 = 10$.

4. (6 points) Consider the initial value problem (IVP) $\frac{dy}{dx} = 3y^{2/3}, \quad y(2) = 0.$

(a) Verify that the constant function $y_1(x) \equiv 0$ is a solution of the IVP.

IF $y(x) = 0$, THEN $\frac{dy}{dx} = 0$. $\quad 0 = 3(0)^{2/3}$

AND $y(2) = 0$. ✓

(b) Verify that $y_2(x) = (x-2)^3$ is also a solution of the IVP.

IF $y(x) = (x-2)^3$, THEN $\frac{dy}{dx} = 3(x-2)^2$

$3(x-2)^2 = 3\left[(x-2)^3\right]^{2/3}$ AND $y(2) = 0$. ✓

(c) Explain why the existence of multiple solutions does not contradict our existence and uniqueness theorem.

$f(x,y) = 3y^{2/3}$.

$f_y(x,y) = 2y^{-1/3}$ WHICH DNE AT $(2,0)$.

THE CONDITIONS FOR UNIQUENESS ARE NOT SATISFIED!

5. (12 points) Analyze each initial value problem to determine which one of these applies.
Do not attempt to solve the equations.

(A) A solution exists, but it is not guaranteed to be unique.

(B) There is a unique solution.

(C) A solution is not guaranteed to exist.

Be sure to show work or explain.

(a) $7xy' + y\sin x = \cos x^2, \quad y(1) = 3$

$f(x,y) = \frac{-y\sin x + \cos x^2}{7x}$

$f_y(x,y) = -\frac{\sin x}{7x}$

BOTH ARE CONTINUOUS AROUND $(1,3)$ $\Rightarrow$ $\boxed{B}$

(b) $y' + x\sqrt[3]{y} = 0, \; y(\pi) = 0$

$f(x,y) = -xy^{1/3}$ ← CONTINUOUS EVERYWHERE

$f_y(x,y) = -\frac{1}{3}xy^{-2/3}$ ← NOT CONT. AT $(\pi,0)$

$\Rightarrow$ $\boxed{A}$

(c) $(y-x)y' = y + x, \quad y(2) = 2$

$f(x,y) = \frac{y+x}{y-x}$ ← NOT CONTINUOUS WHEN $x=y$, AND IN PARTICULAR, NOT AT $(2,2)$

$\Rightarrow$ $\boxed{C}$

6. (8 points) If a population P is growing *exponentially*, then the rate of change of P is proportional to P. Further assume that the population at $t = 0$ is P_0. Write the initial value problem corresponding to this situation and then solve it.

$$\boxed{\frac{dP}{dt} = kP, \quad P(t) = P_0}$$

$$\frac{1}{P}\,dP = k\,dt$$

$$\ln|P| = kt + C_1$$

$$|P| = e^{kt + C_1} = C_2 e^{kt}$$

$$P(t) = C_3 e^{kt}$$

$$P(0) = P_0 \Rightarrow P_0 = C_3 e^0$$

$$\Rightarrow C_3 = P_0$$

$$\boxed{P(t) = P_0 e^{kt}}$$

7. (10 points) Solve the initial value problem: $\quad x^2 y' = y - xy, \quad y(-1) = -1$

$$\frac{dy}{dx} = \frac{(1-x)y}{x^2}$$

$$\frac{1}{y}\,dy = \left(\frac{1-x}{x^2}\right)dx$$

$$\int \frac{1}{y}\,dy = \int \left(\frac{1}{x^2} - \frac{1}{x}\right) dx$$

$$\ln|y| = -\frac{1}{x} - \ln|x| + C_1$$

$$|y| = e^{-1/x} e^{-\ln|x|} e^{C_1}$$

$$|y| = C_2 e^{-1/x}\left(\frac{1}{|x|}\right)$$

BASED ON IC, LET'S ASSUME $x < 0$.

$$|y| = C_2 e^{-1/x}\left(-\frac{1}{x}\right)$$

$$y = C_3 e^{-1/x}\left(\frac{1}{x}\right)$$

$$y(x) = \frac{C_3 e^{-1/x}}{x}$$

$$y(-1) = -1 \Rightarrow -1 = -C_3 e$$

$$C_3 = \frac{1}{e}$$

$$\boxed{y(x) = \frac{e^{-1/x}}{ex}}$$

8. (9 points) Find the value of the constant k so that the equation is exact. Then solve the equation with that choice of k.

$$\underbrace{(2xy^2 + ye^x + 7)}_{M}\, dx + \underbrace{(2x^2y + ke^x - 1)}_{N}\, dy = 0$$

$$\frac{\partial M}{\partial y} = 4xy + e^x = \frac{\partial N}{\partial x} = 4xy + ke^x \Rightarrow \boxed{k = 1}$$

$$F_x = 2xy^2 + ye^x + 7 \Rightarrow F(x,y) = x^2y^2 + ye^x + 7x + g(y)$$

$$F_y = 2x^2y + e^x - 1 \Rightarrow F(x,y) = x^2y^2 + ye^x - y + h(x)$$

$$F(x,y) = x^2y^2 + ye^x + 7x - y$$

$$\boxed{\text{SOLUTION: } x^2y^2 + ye^x + 7x - y = C}$$

9. (13 points) Solve the initial value problem: $(x^2+9)\dfrac{dy}{dx} + xy = x, \quad y(0) = 4$

$$\frac{dy}{dx} + \frac{x}{x^2+9}\, y = \frac{x}{x^2+9}$$

$$\mu(x) = e^{\int \frac{x}{x^2+9}\, dx} = e^{\ln\sqrt{x^2+9}} = \sqrt{x^2+9}$$

ASIDE: $\displaystyle\int \frac{x}{x^2+9}\, dx = \frac{1}{2}\int \frac{1}{u}\, du = \frac{1}{2}\ln|u| = \ln\sqrt{x^2+9}$

$u = x^2+9$

$\frac{1}{2} du = x\, dx$

SOLUTION COMES FROM

$$\sqrt{x^2+9}\; y(x) = \int \sqrt{x^2+9}\, \frac{x}{x^2+9}\, dx$$

$$\int \frac{x}{\sqrt{x^2+9}}\, dx = \frac{1}{2}\int (u)^{-1/2}\, du = u^{1/2} + C$$

$u = x^2+9$

$\frac{1}{2} du = x\, dx$

$$\sqrt{x^2+9}\; y(x) = \sqrt{x^2+9} + C$$

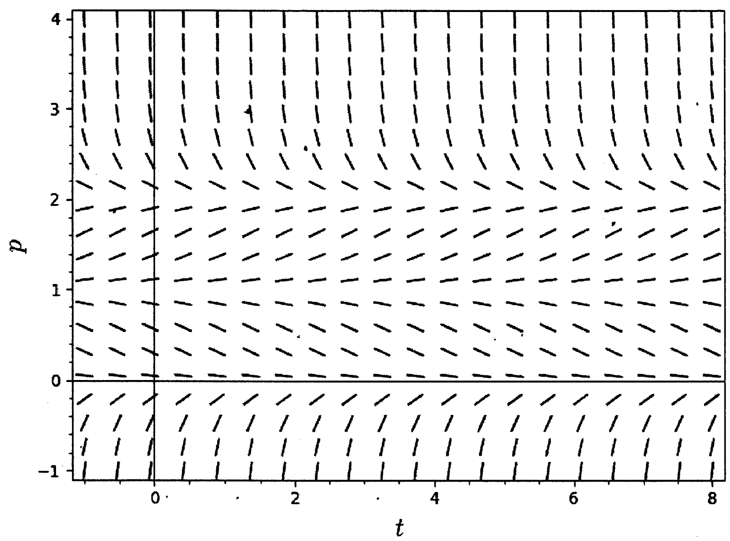
$$y(x) = 1 + \frac{C}{\sqrt{x^2+9}}$$

$$y(0) = 4 \Rightarrow 4 = 1 + \frac{C}{3}$$

$$C = 9$$

$$y(x) = 1 + \frac{9}{\sqrt{x^2+9}}$$

10. (10 points) Consider the differential equation $\frac{dp}{dt} = p(p-1)(2-p)$ for the population p (in thousands) of a certain species at time t. A portion of its slope field is shown below.



(a) What is the slope of the solution curve passing through $(t,p) = (0,3)$?

$$m = \left.\frac{dp}{dt}\right|_{(t,p)=(0,3)} = 3(3-1)(2-3) = \boxed{-6}$$

(b) If the initial population is 3000 (that is, $p(0) = 3$), what can you say about the limiting population $\lim_{t\to\infty} p(t)$?

DECREASES TOWARD 2.

LOOKS LIKE $\lim_{t\to\infty} p(t) = 2$

LIMITING POP IS 2000

(c) If $p(0) = 1.5$, what is $\lim_{t\to\infty} p(t)$?

INCREASES TOWARD 2.

LOOKS LIKE $\lim_{t\to\infty} p(t) = 2$

LIMITING POP IS 2000.

(d) If $p(0) = 0.5$, what is $\lim_{t\to\infty} p(t)$?

DECREASES TOWARD 0.

LOOKS LIKE $\lim_{t\to\infty} p(t) = 0$

LIMITING POP. IS 0.

(e) Can a population of 900 ever increase to 1100?

NO; POP. WILL DECREASE TO 0.

The following problems are due September 17. You must work on your own.

11. (7 points) Consider the initial value problem $y' - y = -x - 1, \quad y(0) = 1$.

(a) Use Euler's method with $h = 0.1$ to estimate $y(0.3)$.

$y(0.3) \approx y_3 = 0.969$

$f(x,y) = y - x - 1$

$y_{n+1} = y_n + h f(x_n, y_n)$

$y_0 = 1$

$x_0 = 0$

$y_1 = 1 + 0.1(1 - 0 - 1) = 1$

$x_1 = 0.1$

$y_2 = 1 + 0.1(1 - 0.1 - 1) = 0.99$

$x_2 = 0.2$

$y_3 = 0.99 + 0.1(0.99 - 0.2 - 1) = 0.969$

(b) Use an Euler's method calculator (or other program) with $h = 0.001$ to approximate $y(0.3)$.

SHOULD GET $y(0.3) \approx y_{300} \approx 0.95034$

(c) Find the exact solution of the IVP.

$\frac{dy}{dx} - y = -x - 1$

$\mu(x) = e^{\int -dx} = e^{-x}$

$e^{-x} y(x) = \int (-x-1)e^{-x}\,dx = (x+1)e^{-x} - \int e^{-x}\,dx$

$u = -x - 1 \qquad du = -dx$

$dv = e^{-x}\,dx \qquad v = -e^{-x}$

$= xe^{-x} + 2e^{-x} + C$

$y(x) = x + 2 + Ce^{x}$

$y(0) = 1$

$\Rightarrow C = -1$

$\boxed{y(x) = x + 2 - e^{x}}$

(d) Use your exact solution to compute $y(0.3)$ and compare it to your approximation in parts (a) and (b).

$y(0.3) = 0.3 + 2 - e^{0.3} = 2.3 - e^{0.3} \approx 0.95014$

THE EXACT SOLUTION IS MUCH CLOSER TO THE APPROXIMATION (b) WITH SMALLER h.

THAT IS WHAT I HAD EXPECTED.

12. (8 points) A tank initially contains 500 kg of salt dissolved in 5000 L of water. A salt solution containing 0.04 kg of salt per liter enters the tank at 20 L/min and is uniformly mixed. The mixed solution leaves the tank at 25 L/min. Let $A(t)$ denote the amount of salt in the tank after t minutes. Set up and solve the appropriate initial value problem to determine $A(t)$. Then find the amount of salt in the tank when the volume has decreased to 2000 L.

$0.04 \frac{kg}{L} \cdot 20 \frac{L}{min}$

$A(0) = 500$

$V(0) = 5000$

$25 \frac{L}{min} \cdot \frac{A(t)}{V(t)} \frac{kg}{L}$

$A(t)$ = AMOUNT OF SALT AT TIME t

$V(t)$ = VOLUME OF SOLUTION AT TIME t

$= \underbrace{5000 - 5t}$

TANK IS EMPTY WHEN $t = 1000$

$$\frac{dA}{dt} = 0.8 - \frac{25A}{5000-5t}, \quad A(0) = 500$$

$$\frac{dA}{dt} + \frac{5}{1000-t} A = 0.8$$

$$\mu(t) = e^{\int \frac{5}{1000-t}dt} = e^{-5 \ln|1000-t|}$$

$$= \frac{1}{(1000-t)^5} \quad \text{ASSUMING } t < 1000$$

$$\frac{1}{(1000-t)^5} A(t) = \int \frac{0.8}{(1000-t)^5} dt$$

$$= \int 0.8(1000-t)^{-5} dt$$

$$= +\frac{0.8}{4}(1000-t)^{-4} + C$$

$$A(t) = 0.2(1000-t) + C(1000-t)^5$$

$A(0) = 500$

$\Rightarrow 500 = 200 + C\, 1000^5$

$\Rightarrow C = \frac{300}{1000^5}$

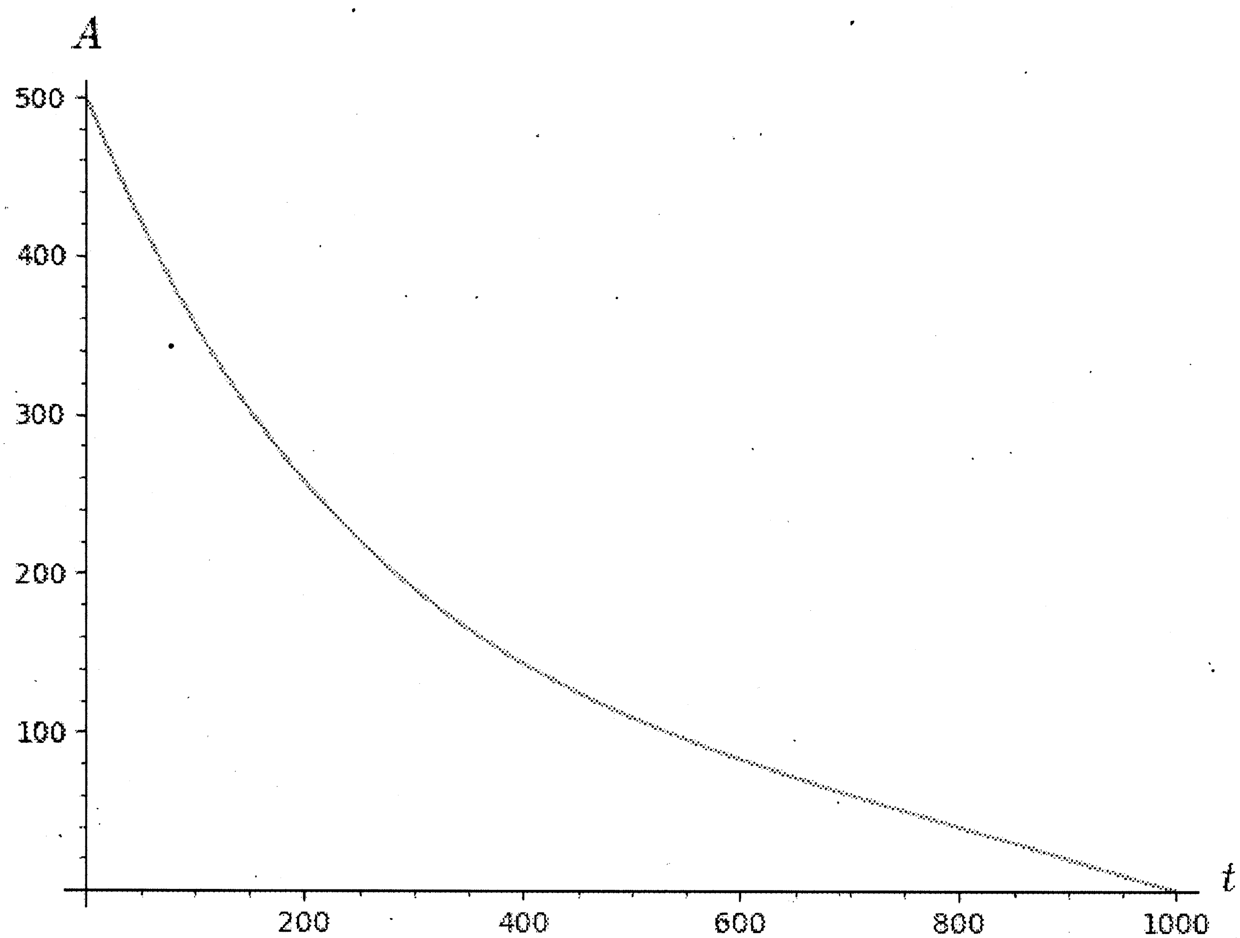
$$A(t) = 0.2(1000-t) + \frac{300}{1000^5}(1000-t)^5$$

$2000 = 5000 - 5t$

$\Rightarrow t = 600$

$$A(600) = \frac{10384}{125} \text{ kg} = 83.072 \text{ kg}$$

★ GRAPH OF $A(t)$ IS ATTACHED.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