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MTH 240 Assignment #8 Key

#1 $(x^2-3)y'' + 2xy' = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$0 = x^2 y'' - 3y'' + 2xy'$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=2}^{\infty} 3n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} 2n a_n x^n$$

Replace
 $n-2$ with n

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} 3(n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} 2n a_n x^n$$

SAME TO
START AT $n=0$

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#1 CONTINUED...

$$= \sum_{n=0}^{\infty} [n(n-1)a_n - 3(n+2)(n+1)a_{n+2} + 2na_n] x^n$$

$$\Rightarrow n(n+1)a_n - 3(n+2)(n+1)a_{n+2} = 0; \quad n=0,1,2,3,\dots$$

$$\Rightarrow a_{n+2} = \frac{n}{3(n+2)} a_n; \quad n=0,1,2,3,\dots$$

a_0 AND a_1 ARE ARBITRARY.

Minimum Radius of Convergence ?

$$y'' + \frac{2x}{x^2-3} y' = 0$$

$$x^2-3=0 \Rightarrow x = \pm\sqrt{3} \quad \text{Singular pts!}$$

$$\text{Distance from center} = \sqrt{3}$$

RADIUS OF CONVERGENCE IS AT LEAST $\sqrt{3}$.

#2

$$5y'' - 2xy' + 10y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$0 = 5y'' - 2xy' + 10y$$

$$= \sum_{n=2}^{\infty} 5n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 10a_n x^n$$

Replace $n-2$ with n SAME TO START WITH $n=0$

$$= \sum_{n=0}^{\infty} 5(n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 10a_n x^n$$

$$= \sum_{n=0}^{\infty} [5(n+2)(n+1) a_{n+2} - 2(n-5) a_n] x^n$$

$$\Rightarrow 5(n+2)(n+1) a_{n+2} - 2(n-5) a_n = 0; \quad n=0, 1, 2, 3, \dots$$

$$\Rightarrow a_{n+2} = \frac{2(n-5)}{5(n+2)(n+1)} a_n; \quad n=0, 1, 2, 3, \dots$$

a_0 & a_1 ARE ARBITRARY.

Min. RADIUS OF CONVERGENCE ?

$$y'' - \frac{2x}{5} y' + \frac{10}{5} y = 0$$

COEFFICIENTS ARE ANALYTIC EVERYWHERE.

No singular pts.

RADIUS OF CONVERGENCE IS ∞ .

#3 $y'' - x^2 y' - 3xy = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$0 = y'' - x^2 y' - 3xy$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^{n+1} - \sum_{n=0}^{\infty} 3 a_n x^{n+1}$$

REPLACE
n-2 WITH n

REPLACE n+1
WITH n

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n - \sum_{n=1}^{\infty} 3 a_{n-1} x^n$$

$$= 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n - 3a_0 x - \sum_{n=2}^{\infty} 3a_{n-1} x^n$$

$$= 2a_2 + (6a_3 - 3a_0)x + \sum_{n=2}^{\infty} [(n+2)(n+1) a_{n+2} - (n-1) a_{n-1} - 3a_{n-1}] x^n$$

$$\Rightarrow a_2 = 0, \quad 6a_3 = 3a_0, \quad (n+2)(n+1) a_{n+2} - (n+2) a_{n-1} = 0; \quad n=2,3,4,\dots$$

$$\Rightarrow a_2 = 0, \quad a_3 = \frac{1}{2} a_0, \quad a_{n+2} = \frac{1}{n+1} a_{n-1}; \quad n=2,3,4,\dots$$

a_0 & a_1 ARE ARBITRARY.

MIN. RADIUS OF CONVERGENCE ?

THERE ARE NO SINGULAR PTS.

RADIUS OF CONVERGENCE IS ∞ .

#4 $x^3 y' = 2y$

a) Assume $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$$0 = x^3 y' - 2y$$

$$= \sum_{n=1}^{\infty} n a_n x^{n+2} - \sum_{n=0}^{\infty} 2 a_n x^n$$

Replace
 $n+2$ with
 n

$$= \sum_{n=3}^{\infty} (n-2) a_{n-2} x^n - \sum_{n=0}^{\infty} 2 a_n x^n$$

$$= \sum_{n=3}^{\infty} (n-2) a_{n-2} x^n - 2a_0 - 2a_1 x - 2a_2 x^2 - \sum_{n=3}^{\infty} 2 a_n x^n$$

$$\Rightarrow a_0 = a_1 = a_2 = 0, \quad (n-2) a_{n-2} - 2a_n = 0; \quad n = 3, 4, 5, \dots$$

$$a_n = \frac{n-2}{2} a_{n-2}; \quad n = 3, 4, 5, \dots$$

↑ All a_n 's are zero!

We found $y(x) = 0$. This is a particular solution,
but not the general solution.

$$b) \quad y' - \frac{2}{x^3} y = 0$$

$$\mu(x) = e^{\int -\frac{2}{x^3} dx} = e^{1/x^2}$$

$$e^{1/x^2} y(x) = \int 0 dx = C$$

$$y(x) = \frac{C}{e^{1/x^2}}$$

c) We should not have expected a power series centered at $x=0$ to provide a solution because $x=0$ is a singular point.

$$\#5 \quad a) \quad y'' + \frac{7x}{x+4} y' - \frac{5(x+2)}{x+4} y = 0$$

$x = -4$ is the only singular pt.

Distance from $x=0$ to $x=-4$ is 4 units.

Radius of convergence is at least 4.

$$b) \quad y'' + \frac{8}{x^2+1} y' + \frac{x^2-1}{x^2+1} y = 0$$

Singular pts are $x = \pm i$.

Distance from $x=0$ to $x = \pm i$ is 1 unit.

Radius of convergence is at least 1.