Math 236 - Test 3

Name _____

April 15, 2026

Score _____

Show all work to receive full credit. Supply explanations when necessary. You may use your calculator to obtain any RREF.

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1. (8 points) Suppose that D is a diagonal matrix[1] with nonzero diagonal entries $d_1, d_2, \ldots d_n$.

   (a) Argue that D is invertible.

   (b) Describe D^{-1} and explain how you know.

2. (4 points) Show that $R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is invertible for any real θ. Find R^{-1}.

[1] A diagonal matrix is a square matrix with zeros everywhere except possibly along the main diagonal.

3. (8 points) Suppose that H and G are $n \times n$ invertible matrices.

   (a) Use our properties of rank to argue that the product HG is invertible.

   (b) Prove that $(HG)^{-1} = G^{-1}H^{-1}$.

4. (8 points) Show that H has infinitely many right inverses, but that it has no left inverse.

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

5. (4 points) Suppose B is a basis for a 4-dimensional vector space. What is the change-of-basis matrix with respect to B, B?

6. (15 points) Consider the bases $B, D \subseteq \mathcal{P}_2$.

$$B = \langle 1, x, x^2 \rangle, \qquad D = \langle 1 + x - x^2, 1 + x^2, 1 + x \rangle$$

(a) Find the change-of-basis matrix with respect to B, D.

(b) Let $p(x) = 8 - 7x + 3x^2$. Find $\text{Rep}_D(p(x))$.

(c) Use any approach to find the change-of-basis matrix with respect to D, B.

7. (8 points) Consider the vector space of real-valued functions with basis $B = \langle \sin x, \cos x \rangle$. It can be shown (don't bother) that $D = \langle 2\sin x + \cos x, 3\cos x \rangle$ is also a basis for this space. Find the change-of-basis matrix with respect to B, D.

8. (12 points) Consider the basis for $\mathbb{R}^3$ shown below:

$$B = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\rangle.$$

Use the Gram-Schmidt process to find the corresponding orthogonal basis. You need not normalize.

9. (7 points) Suppose that $B = \{\vec{\beta}_1, \vec{\beta}_2, \ldots, \vec{\beta}_n\}$ is a set of nonzero mutually orthogonal vectors. Prove that the set is linearly independent.

10. (4 points) Is the following statement true or false? Briefly justify your answer.

Every finite-dimensional vector space has an orthonormal basis.

11. (4 points) Suppose S is an $n \times n$ nonsingular matrix, and suppose that A and B are $n \times n$ matrices such that $SAS^{-1} = B$. Prove that $\det(A) = \det(B)$.

12. (6 points) Suppose A is an $n \times n$ matrix with $\det(A) = 3$.

(a) Compute $\det(A^4)$.

(b) Compute $\det(2A)$.

(c) Compute $\det(A^T)$.

13. (8 points) Let $H = \begin{pmatrix} 2 & -2 & -2 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix}$. Determine H^{-1} by using the matrix adjoint.

14. (4 points) Use Cramer's rule to solve the system: $\begin{array}{rcrcl} 3x & + & 4y & = & 11 \\ 5x & - & 9y & = & 6 \end{array}$