Math 236 - Test 3

April 15, 2026

Name key

Score

Show all work to receive full credit. Supply explanations when necessary. You may use your calculator to obtain any RREF.

1. (8 points) Suppose that D is a diagonal matrix[1] with nonzero diagonal entries $d_1, d_2, \ldots d_n$.

   (a) Argue that D is invertible.

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & 0 \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 \cdots & 0 & d_n \end{pmatrix}$$

D IS INVERTIBLE BECAUSE

$$\det(D) = d_1 \cdot d_2 \cdots d_n \neq 0.$$

-OR-

THE COLUMN (ROWS) OF D ARE CLEARLY INDEPENDENT.

   (b) Describe D^{-1} and explain how you know.

$$D^{-1} = \begin{pmatrix} \frac{1}{d_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & 0 \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{d_n} \end{pmatrix} = \operatorname{diag}\left(\frac{1}{d_1}, \frac{1}{d_2}, \ldots, \frac{1}{d_n}\right)$$

TO VERIFY THIS, JUST DO THE MULTIPLICATION:

$$DD^{-1} = I_n.$$

2. (4 points) Show that $R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is invertible for any real θ. Find R^{-1}.

$$\det(R) = \cos^2\theta + \sin^2\theta = 1 \neq 0 \Rightarrow R \text{ IS INVERTIBLE.}$$

FROM THE FORMULA FOR THE INVERSE OF A 2x2 MATRIX, $R^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

[1] A diagonal matrix is a square matrix with zeros everywhere except possibly along the main diagonal.

3. (8 points) Suppose that H and G are $n \times n$ invertible matrices.

(a) Use our properties of rank to argue that the product HG is invertible.

$\text{rank}(HG) = \text{rank}(H)$ SINCE G IS INVERTIBLE (NONSING, RANK n)

AND $\text{rank}(H) = n$ SINCE H IS INVERTIBLE (NONSING, RANK n).

$\therefore$ $\text{rank}(HG) = n$ AND HG IS INVERTIBLE / NONSING / RANK n.

(b) Prove that $(HG)^{-1} = G^{-1}H^{-1}$.

$$(HG)^{-1}HG = I \overset{\text{RIGHT MULT BY } G^{-1}}{\Rightarrow} (HG)^{-1}H = G^{-1} \overset{\text{RIGHT MULT BY } H^{-1}}{\Rightarrow} (HG)^{-1} = G^{-1}H^{-1} \checkmark$$

4. (8 points) Show that H has infinitely many right inverses, but that it has no left inverse.

$$H = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

RIGHT INVERSE ...

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$2a + e = 1 \qquad 2b + f = 0$

$e = 0 \qquad f = 1$

$a = \frac{1}{2}, \ b = -\frac{1}{2}$

$e = 0, \ f = 1$

$c = d =$ ANY NUMBER $\Rightarrow$ INF MANY CHOICES FOR RIGHT INVERSE.

LEFT INVERSE ...

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$0c + 0d = 0 \neq 1$

NOT POSSIBLE.

COULD ALSO USE RANK:

$\text{rank}(AH) \neq 3$ FOR ANY A.

5. (4 points) Suppose B is a basis for a 4-dimensional vector space. What is the change-of-basis matrix with respect to B, B?

$$\left(\text{Rep}_B(\vec{\beta}_1) \,\middle|\, \text{Rep}_B(\vec{\beta}_2) \,\middle|\, \text{Rep}_B(\vec{\beta}_3) \,\middle|\, \text{Rep}_B(\vec{\beta}_4) \right)_{B,B}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{B,B}.$$

6. (15 points) Consider the bases $B, D \subseteq \mathcal{P}_2$.

$$B = \langle 1, x, x^2 \rangle, \qquad D = \langle 1 + x - x^2, 1 + x^2, 1 + x \rangle$$

(a) Find the change-of-basis matrix with respect to B, D.

$$\mathrm{Rep}_{B,D}(id) = \begin{pmatrix} \mathrm{Rep}_D(1) & \mathrm{Rep}_D(x) & \mathrm{Rep}_D(x^2) \end{pmatrix}_{B,D} = \boxed{\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix}_{B,D}}$$

$a+b+c=1$
$a \quad +c=0$
$-a+b \quad =0$

$b+c=0$
$a=1.$ $c=-1$
$b=1$

$a+b+c=0$
$a \quad +c=1$
$-a+b \quad =0$

$b+c=1$
$a=-1.$
$c=2$
$b=-1$

$a+b+c=0$
$a \quad +c=0$
$-a+b \quad =1$

$b+c=1$
$a=-1$
$b=0$
$c=1$

(b) Let $p(x) = 8 - 7x + 3x^2$. Find $\mathrm{Rep}_D(p(x))$.

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix}_{B,D} \begin{pmatrix} 8 \\ -7 \\ 3 \end{pmatrix}_B = \boxed{\begin{pmatrix} 12 \\ 15 \\ -19 \end{pmatrix}_D}$$

CHECK:
$12(1+x-x^2)+15(1+x^2)$
$-19(1+x)$
$= 8-7x+3x^2$ ✓

(c) Use any approach to find the $\underbrace{\text{change-of-basis matrix with respect to}}_{\text{INVERSE OF MATRIX IN (a).}}$ D, B.

$$\mathrm{Rep}_{D,B}(id) = \begin{pmatrix} \mathrm{Rep}_B(1+x-x^2) & \mathrm{Rep}_B(1+x^2) & \mathrm{Rep}_B(1+x) \end{pmatrix}_{D,B}$$

$$= \boxed{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}_{D,B}}$$

CHECK: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix}$
$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ✓

7. (8 points) Consider the vector space of real-valued functions with basis $B = \langle \sin x, \cos x \rangle$. It can be shown (don't bother) that $D = \langle 2\sin x + \cos x, 3\cos x \rangle$ is also a basis for this space. Find the change-of-basis matrix with respect to B, D.

$$\mathrm{Rep}_{B,D}(id) = \begin{pmatrix} \mathrm{Rep}_D(\sin x) & \Big| & \mathrm{Rep}_D(\cos x) \end{pmatrix}_{B,D} = \boxed{\begin{pmatrix} 1/2 & 0 \\ -1/6 & 1/3 \end{pmatrix}_{B,D}}$$

$$\sin x = a(2\sin x + \cos x) + b(3\cos x)$$

$2a = 1 \qquad a = \frac{1}{2}$

$a + 3b = 0 \qquad b = -\frac{1}{6}$

$$\cos x = a(2\sin x + \cos x) + b(3\cos x)$$

$2a = 0$

$a + 3b = 1$

8. (12 points) Consider the basis for $\mathbb{R}^3$ shown below:

$$B = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\rangle.$$

Use the Gram-Schmidt process to find the corresponding orthogonal basis. You need not normalize.

$$\vec{k}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{k}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{\vec{k}_1 \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}{\vec{k}_1 \cdot \vec{k}_1} \vec{k}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1 \\ 1/2 \end{pmatrix}$$

$$\vec{k}_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \frac{\vec{k}_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}}{\vec{k}_1 \cdot \vec{k}_1} \vec{k}_1 - \frac{\vec{k}_2 \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}}{\vec{k}_2 \cdot \vec{k}_2} \vec{k}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \frac{3}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1/2}{3/2} \begin{pmatrix} -1/2 \\ 1 \\ 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{3}{2} + \frac{1}{6} \\ 0 - 0 - \frac{1}{3} \\ 2 - \frac{3}{2} - \frac{1}{6} \end{pmatrix} = \begin{pmatrix} -1/3 \\ -1/3 \\ 1/3 \end{pmatrix}$$

$$K = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1 \\ 1/2 \end{pmatrix}, \begin{pmatrix} -1/3 \\ -1/3 \\ 1/3 \end{pmatrix} \right\rangle$$

9. (7 points) Suppose that $B = \{\vec{\beta}_1, \vec{\beta}_2, \ldots, \vec{\beta}_n\}$ is a set of nonzero mutually orthogonal vectors. Prove that the set is linearly independent.

SUPPOSE $c_1 \vec{\beta}_1 + c_2 \vec{\beta}_2 + \cdots + c_n \vec{\beta}_n = \vec{0}$.

FOR EACH $k = 1, 2, 3, \ldots, n$,

$$\vec{\beta}_k \cdot \underbrace{(c_1 \vec{\beta}_1 + c_2 \vec{\beta}_2 + \cdots + c_n \vec{\beta}_n)}_{\vec{0}} = c_k \vec{\beta}_k \cdot \vec{\beta}_k$$

$$= c_k \|\vec{\beta}_k\|^2 = 0$$

$\|\vec{\beta}_k\|^2$ CANNOT BE ZERO, SINCE $\vec{\beta}_k \neq \vec{0}$.

$\therefore c_k = 0$.

10. (4 points) Is the following statement true or false? Briefly justify your answer.

Every finite-dimensional vector space has an orthonormal basis.

TRUE. EVERY VECTOR SPACE HAS A BASIS. USE GRAM-SCHMIDT TO FORM ORTHONORMAL BASIS.

* THE BASIS FOR THE 0-dim SPACE IS THE EMPTY SET. SINCE IT'S EMPTY, EVERYTHING IN IT IS AUTOMATICALLY NORMALIZED.

11. (4 points) Suppose S is an $n \times n$ nonsingular matrix, and suppose that A and B are $n \times n$ matrices such that $SAS^{-1} = B$. Prove that $\det(A) = \det(B)$.

$$\det(B) = \det(SAS^{-1}) = \det(S)\det(A)\det(S^{-1})$$

$$= \cancel{\det(S)}\det(A)\frac{1}{\cancel{\det(S)}}$$

$$= \det(A).$$

12. (6 points) Suppose A is an $n \times n$ matrix with $\det(A) = 3$.

(a) Compute $\det(A^4)$.

$$= \left[\det(A)\right]^4 = \boxed{81}$$

(b) Compute $\det(2A)$.

$$\boxed{2^n \cdot 3}$$ ONE FACTOR OF 2 FOR EACH ROW.

(c) Compute $\det(A^T)$.

$$= \boxed{3} = \det(A)$$

13. (8 points) Let $H = \begin{pmatrix} 2 & -2 & -2 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix}$. Determine H^{-1} by using the matrix adjoint.

$$\det(H) = -2(2-1) + (-2+2) = -2$$

COFACTOR MATRIX $= \begin{pmatrix} -1 & -1 & 1 \\ -2 & 0 & -2 \\ -2 & -2 & 0 \end{pmatrix}$

$$H^{-1} = \frac{-1}{2}\begin{pmatrix} -1 & -2 & -2 \\ -1 & 0 & -2 \\ 1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 1 & 1 \\ 1/2 & 0 & 1 \\ -1/2 & 1 & 0 \end{pmatrix}$$

14. (4 points) Use Cramer's rule to solve the system:
$$\begin{aligned} 3x + 4y &= 11 \\ 5x - 9y &= 6 \end{aligned}$$

$$x = \frac{\begin{vmatrix} 11 & 4 \\ 6 & -9 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 5 & -9 \end{vmatrix}} = \frac{-99-24}{-27-20} = \frac{123}{47}$$

$$y = \frac{\begin{vmatrix} 3 & 11 \\ 5 & 6 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 5 & -9 \end{vmatrix}} = \frac{18-55}{-47} = \frac{37}{47}$$