Math 236 - Test 2

March 11, 2026

Name ______________________________

Score ____________

Show all work to receive full credit. Supply explanations when necessary. You may use your calculator to obtain any RREF.

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1. (10 points) Consider the following set of matrices in $\mathcal{M}_{2\times 2}$.

$$S = \left\{ A_1 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 4 & -2 \\ 2 & -2 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, A_4 = \begin{pmatrix} 5 & 0 \\ 1 & -1 \end{pmatrix} \right\}$$

   (a) Show that the set S is linearly dependent set.

   (b) Write any one of the matrices in S as a linear combination of the others.

2. (8 points) Find a basis for the row space, a basis for the column space, and the rank of the matrix A.

$$A = \begin{pmatrix} 1 & 4 & 5 & 1 & 0 \\ -2 & 0 & 6 & -1 & -6 \\ 3 & 1 & -7 & 3 & 11 \end{pmatrix}$$

3. (8 points) Let W be the subspace of $\mathcal{P}_3$ defined by

$$W = \{ax^3 + bx^2 + cx + d : 2a + b + c = 0 \text{ and } 2b + d = 0\}.$$

(a) Find a basis for W.

(b) What is the dimension of W?

(c) Represent $p(x) = -2x^3 - 6x^2 + 10x + 12$ in terms of your basis.

4. (4 points) A matrix has 5 rows and 9 columns. Which set must be dependent, its set of rows or its set of columns? Carefully explain your reasoning.

5. (8 points) Let $W = \text{span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\})$, where

$$\vec{v}_1 = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 4 \\ -9 \\ -1 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} -2 \\ 7 \\ 3 \end{pmatrix}.$$

(a) Briefly explain why the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, and $\vec{v}_4$ cannot be linearly independent.

(b) Find a basis for span(W).

6. (8 points) Suppose $h : \mathcal{P}_2 \to \mathcal{M}_{2\times 2}$ is a homomorphism with

$$h(x^2 + 2x + 1) = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}, \qquad h(x^2 - 3) = \begin{pmatrix} -1 & 5 \\ 0 & -3 \end{pmatrix}, \qquad h(x + 1) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Determine $h(5x^2 - x - 12)$.

7. (10 points) Suppose $\vec{x}_1, \vec{x}_2, \dots \vec{x}_k$ are a linearly independent vectors in the vector space V. Also suppose that $f : V \to W$ is an isomorphism. Prove that $f(\vec{x}_1), f(\vec{x}_2), \dots f(\vec{x}_k)$ are linearly independent in W.

Follow up: If f is a homomorphism, but not an isomorphism, the result is no longer true. What goes wrong in your proof?

8. (6 points) Suppose $g : U \to V$ and $f : V \to W$ are isomorphisms. Show that the composition of functions, $f \circ g : U \to W$, is one-to-one and onto. (Recall that $(f \circ g)(x)$ means $f(g(x))$.)

9. (10 points) Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ 2x \end{pmatrix}.$$

Show that f is an isomorphism.

10. (8 points) Define the homomorphism $h : \mathcal{P}_2 \to \mathbb{R}^3$ by

$$h(ax^2 + bx + c) = \begin{pmatrix} a - c \\ 0 \\ b + 2c \end{pmatrix}.$$

(a) Before doing any work on this problem, determine the sum of the rank and nullity of h. Very briefly say how you know.

(b) Determine a basis for the range space of h.

(c) Determine a basis for the null space of h.

The following problems are due March 23. You must work on your own. If you are not familiar with matrix multiplication, acquaint yourself with it before continuing. If necessary, a simple Google search should provide sufficient details.

11. (2 points) Just for a quick warm-up, compute the product AB by hand. (After you finish your computation, you may check your result with a computer or calculator.)

$$A = \begin{pmatrix} 2 & 4 & -5 \\ -3 & 4 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 7 \\ 2 & 3 \\ -4 & 5 \end{pmatrix}$$

12. (2 points) Suppose $A \in \mathcal{M}_{m\times n}$ and $b \in \mathbb{R}^n$. The matrix-vector product Ab can be computed by using usual matrix multiplication, treating the vector b as an $n \times 1$ matrix.

    (a) Compute Ab when $A = \begin{pmatrix} 2 & 1 & 3 \\ 9 & 0 & -2 \\ 2 & 5 & -3 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$.

    (b) It is often convenient to think of the product Ab as a linear combination of the columns of A with the coefficients from b. Use this approach to recompute Ab from above.

13. (4 points) Consider the matrix $A = \begin{pmatrix} 1 & 0 & 3 & 1 & -2 \\ 2 & 3 & -9 & 1 & 0 \\ -1 & 5 & -28 & 2 & -10 \\ 3 & -2 & 19 & 4 & -10 \end{pmatrix}$.

(a) Compute the RREF of A (use technology!). If applicable, remove any rows that consist entirely of zeros, and call the remaining matrix R.

(b) Use your RREF to determine the linearly independent columns of the original matrix A. Put those columns into a matrix and call it C.

(c) The matrix product CR must be defined. Explain how you know.

(d) Compute CR and compare it to A.

14. (12 points) Each of the elementary row operations that we use in Gauss-Jordan elimination correspond to matrix multiplication. The matrices that "perform" the elementary row operations are called *elementary matrices*, and there are three general types, one for each row operation. For example, if $A \in \mathcal{M}_{3\times n}$ and

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

then the product EA is the matrix A with its 2nd and 3rd rows swapped. Similarly, if

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

then the product EA is the matrix A with it 3rd row replaced by -2 times the original 3rd row. Finally, when multiplied on the left of A, the matrix

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix}$$

would leave rows 1 and 2 unchanged, but would replace the 3rd row of A with 5 times the 1st row plus the 3rd row.

Experiment with these types of matrix multiplications! Continue with this problem after you feel comfortable. Feel free to use a computer or calculator to check all of your multiplications.

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(a) Let $A = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix}$.

By hand, convert A to RREF, and keep track (in order) of the sequence of elementary matrices that correspond to your row operations. When all is said and done, you should have elementary matrices $E_1, E_2, \ldots E_k$ with the property that

$$E_k \cdots E_2 E_1 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Continued $\longrightarrow$

(b) Recall that each elementary row operation is reversible. Go back to each of your matrices from above and determine the elementary matrix that reverses (undoes) each row operation. Use the notation E'_j for the elementary matrix that "undoes" E_j.

(c) Use a computer or calculator to check that $E'_1 E'_2 \cdots E'_k = A$.

Congratulations! You have just shown, by example, that every nonsingular matrix is a product of elementary matrices. This will be a very useful idea in the future.