Math 236 - Assignment 9

April 8, 2026

Name ____________________

Score ______________

Show all work to receive full credit. Supply explanations when necessary. This assignment is due April 15.

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1. Suppose that $B = \{\vec{\beta}_1, \vec{\beta}_2, \ldots, \vec{\beta}_n\}$ is a set of nonzero mutually orthogonal vectors. Prove that the set is linearly independent.

   Solution

   Assume $\vec{\beta}_1, \vec{\beta}_2, \ldots, \vec{\beta}_n$ are nonzero and mutually orthogonal. Now suppose

   $$c_1\vec{\beta}_1 + s_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n = \vec{0}.$$

   For each $k = 1, 2, \ldots, n$, compute a dot product on each side:

   $$\vec{\beta}_k \cdot (c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n) = \vec{\beta}_k \cdot \vec{0} = 0.$$

   Distribute and use orthogonality to get $c_k\vec{\beta}_k \cdot \vec{\beta}_k = 0$ or $c_k\|\vec{\beta}_k\|^2 = 0$.

   Since $\|\vec{\beta}_k\| \neq 0$, we must have $c_k = 0$. This is true for each k.

2. Use any method to compute the determinant by hand.

   $$A = \begin{pmatrix} 2 & 4 & -6 \\ 5 & 1 & 3 \\ 2 & -6 & 1 \end{pmatrix}$$

   Solution

   Using the Laplace expansion along the 1st row:

   $$D = 2(1 + 18) - 4(5 - 6) + (-6)(-30 - 2) = 234.$$

3. Use Gaussian elimination to compute the determinant by hand.

   $$B = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 1 & 0 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 1 & 1 & -1 & 1 \end{pmatrix}$$

   Solution

   Here are Sage steps that reduce the matrix to echelon form:

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B=matrix([[1,-1,1,2],[1,0,1,3],[0,0,2,4],[1,1,-1,1]])
B.add_multiple_of_row(1,0,-1)
B.add_multiple_of_row(3,0,-1)
B.add_multiple_of_row(3,1,-2)
B.add_multiple_of_row(3,2,1)
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The corresponding echelon form is

$$\begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

None of these row operations change the value of $\det(B)$. Therefore, $\det(B) = 2$.

4. Use Cramer's rule to solve the following system of equations.

$$\begin{array}{rcrcl} 7x & + & 3y & = & 12 \\ 3x & - & 9y & = & 2 \end{array}$$

Solution

$$x = \frac{\begin{vmatrix} 12 & 3 \\ 2 & -9 \end{vmatrix}}{\begin{vmatrix} 7 & 3 \\ 3 & -9 \end{vmatrix}} = \frac{-114}{-72} = \frac{19}{12}, \qquad y = \frac{\begin{vmatrix} 7 & 12 \\ 3 & 2 \end{vmatrix}}{\begin{vmatrix} 7 & 3 \\ 3 & -9 \end{vmatrix}} = \frac{-22}{-72} = \frac{11}{36}$$

5. Determine all real numbers x that make this matrix singular?

$$A = \begin{pmatrix} 1-x & 3 & 3 \\ -3 & -5-x & -3 \\ 3 & 3 & 1-x \end{pmatrix}$$

Solution

Using the Laplace expansion along the 1st row,

$$\det(A) = (1-x)[(-5-x)(1-x)+9] - 3[-3(1-x)+9] + 3[-9-3(-5-x)]$$
$$= -(x-1)(x^2+4x+4) = -(x-1)(x+2)^2.$$

It follows that $x = 1$ or $x = -2$ (multiplity 2).

6. Let $A = \begin{pmatrix} 3 & 2 \\ 1 & -5 \end{pmatrix}$. Find all real or complex numbers x that make the matrix $A - xI$ singular.

Solution

$$\det(A - xI) = \begin{vmatrix} 3-x & 2 \\ 1 & -5-x \end{vmatrix} = (x-3)(x+5) - 2 = x^2 + 2x - 17 = (x+1)^2 - 18$$

$$\det(A - xI) = 0 \quad \Longrightarrow \quad x = -1 \pm \sqrt{18}$$

7. For which values of k does this system have a unique solution?

$$\begin{array}{ccccccccc} x & & & + & z & - & w & = & 2 \\ & & y & - & 2z & & & = & 3 \\ x & & & + & kz & & & = & 4 \\ & & & & z & - & w & = & 2 \end{array}$$

Solution

The system has a unique solution when the determinant of the coefficient matrix is nonzero. Expanding down the 2nd column,

$$\begin{vmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 0 \\ 1 & 0 & k & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & k & 0 \\ 0 & 1 & -1 \end{vmatrix} = -k.$$

The system will have a unique solution for any nonzero k.

8. What is the determinant of each kind of elementary matrix?

Solution

- A "multiply row k by α and add to row j" matrix is an identity matrix with single off-diagonal nonzero entry. Its determinant is 1.
- A "row swap" matrix is an identity matrix with two rows swapped. Its determinant is -1.
- A "multiply row by nonzero constant" matrix is an identity matrix with a single diagonal entry replaced by the nonzero row multiplier k. Its determinant is k.

9. Use the Laplace expansion to compute the determinant by hand. Expand over whichever row or column is most convenient.

$$A = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 1 & 1 \\ 3 & -1 & 0 \end{pmatrix}$$

Solution

Using the Laplace expansion down the 3rd column,

$$\det(A) = (-1)(-1 - 15) = 16.$$

10. Find the inverse by using the matrix adjoint.

$$A = \begin{pmatrix} 1 & 4 & 3 \\ -1 & 0 & 4 \\ 1 & 8 & 9 \end{pmatrix}$$

Solution

First, let's find $\det(A)$ by expanding down the 2nd row:

$$\det(A) = (36 - 24) + (-4)(8 - 4) = -4.$$

The cofactor matrix for A is given by

$$\begin{pmatrix} -32 & 13 & -8 \\ -12 & 6 & -4 \\ 16 & -7 & 4 \end{pmatrix}.$$

It follows that

$$A^{-1} = -\frac{1}{4}\begin{pmatrix} -32 & -12 & 16 \\ 13 & 6 & -7 \\ -8 & -4 & 4 \end{pmatrix} = \begin{pmatrix} 8 & 3 & -4 \\ -13/4 & -3/2 & 7/4 \\ 2 & 1 & -1 \end{pmatrix}.$$