

# Math 236 - Assignment 8

April 1, 2026

Name \_\_\_\_\_

Score \_\_\_\_\_

Show all work to receive full credit. Supply explanations when necessary. This assignment is due April 8.

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1. A *permutation matrix* is a square matrix that has exactly one 1 in each row and column and 0's everywhere else. Experiment with multiplications of random matrices by permutation matrices (on either side), and try to predict their effect. Argue that the inverse of any permutation matrix is its transpose. (You need not give a formal proof, just a compelling argument.)

## Solution

Suppose  $P$  is a permutation matrix with rows  $P_1, P_2, \dots, P_n$ . Then  $P^T$  is the matrix with columns  $P_1^T, P_2^T, \dots, P_n^T$ . Each row  $P_k$  and column  $P_k^T$  has a single 1 (in the same position) and 0's elsewhere. No other row or column has a 1 in the same position. It follows that

$$P_i \cdot P_j^T = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

from which it follows that  $PP^T = I$ .

(Another approach is to recognize that  $P$  is an orthogonal matrix, from which the result follows immediately.)

2. For an invertible matrix  $A$  and a positive integer  $k$ , prove that  $(A^k)^{-1} = (A^{-1})^k$ . (Technically, this proof requires a technique called induction. We will eventually talk about induction. For now, to complete the proof, simply write out what  $A^k$  and  $(A^{-1})^k$  mean, and show that their product is  $I$ .)

## Solution

First notice that

$$(A^k)(A^{-1})^k = A \cdot A \cdots A \cdot A^{-1} \cdot A^{-1} \cdots A^{-1},$$

where each product occurs  $k$  times. Using  $AA^{-1} = I$  and simplifying from the inside out, we get  $(A^k)(A^{-1})^k = I$ .

Also, we have

$$(A^{-1})^k A^k = A^{-1} \cdot A^{-1} \cdots A^{-1} \cdot A \cdot A \cdots A,$$

where each product occurs  $k$  times. Using  $A^{-1}A = I$  and simplifying from the inside out, we get  $(A^{-1})^k(A^k) = I$ .

Therefore,  $(A^{-1})^k$  is both a right and left inverse of  $A^k$ , and it follows that  $(A^{-1})^k = (A^k)^{-1}$ .

3. Suppose that  $A$  and  $B$  are  $n \times n$  matrices. Prove that  $AB = I$  if and only if  $BA = I$ . (Helpful hint: From the fact that  $\text{rank}(AB) = \text{rank}(BA) = n$ , it follows that  $\text{rank}(A) = \text{rank}(B) = n$ .)

Solution

Suppose  $AB = I$ . Then  $\text{rank}(AB) = n$ , and by our rank inequalities,

$$n = \text{rank}(AB) \leq \text{rank}(A) \leq n$$

and

$$n = \text{rank}(AB) \leq \text{rank}(B) \leq n.$$

It follows that  $\text{rank}(A) = \text{rank}(B) = n$ , and  $A$  and  $B$  are invertible.

Since  $A$  and  $B$  are invertible, we have

$$AB = I \iff A^{-1}AB = A^{-1}I \iff B = A^{-1} \iff BA = A^{-1}A = I.$$

Therefore

$$AB = I \iff BA = I.$$

4. Find the change of basis matrix for  $B, D \subseteq \mathbb{R}^2$ .

$$B = \left\langle \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \end{pmatrix} \right\rangle, \quad D = \left\langle \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle$$

Solution

Some details are omitted.

$$\begin{aligned} \text{Rep}_{B,D}(id) &= \left( \text{Rep}_D \left( \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right) \quad \text{Rep}_D \left( \begin{pmatrix} 1 \\ -3 \end{pmatrix} \right) \right) \\ &= \begin{pmatrix} -3/2 & 7/4 \\ 5/2 & -5/4 \end{pmatrix}_{B,D} \end{aligned}$$

5. Find the change of basis matrix for  $B, D \subseteq \mathcal{P}_1$ .

$$B = \langle 1 + x, 1 - x \rangle, \quad D = \langle 2x, 1 - 2x \rangle$$

Solution

Some details are omitted.

$$\begin{aligned} \text{Rep}_{B,D}(id) &= \left( \text{Rep}_D(1 + x) \quad \text{Rep}_D(1 - x) \right) \\ &= \begin{pmatrix} 3/2 & 1/2 \\ 1 & 1 \end{pmatrix}_{B,D} \end{aligned}$$

6. Find the change of basis matrix for  $B, D \subseteq \mathcal{P}_2$ .

$$B = \langle 1, x, x^2 \rangle, \quad D = \langle x^2, 1, x \rangle$$

Some details are omitted.

$$\begin{aligned} \text{Rep}_{B,D}(\text{id}) &= (\text{Rep}_D(1) \quad \text{Rep}_D(x) \quad \text{Rep}_D(x^2)) \\ &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}_{B,D} \end{aligned}$$

7. Perform the Gram-Schmidt process on this basis for  $\mathbb{R}^3$ :

$$\left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \right\rangle.$$

Solution

$$\begin{aligned} \vec{\kappa}_1 &= (1, 2, 3)^T \\ \vec{\kappa}_2 &= (2, 1, -3)^T - \frac{\vec{\kappa}_1 \cdot (2, 1, -3)^T}{\vec{\kappa}_1 \cdot \vec{\kappa}_1} \vec{\kappa}_1 = (2, 1, -3)^T + \frac{5}{14} (1, 2, 3)^T = (33/14, 12/7, -27/14)^T \\ \vec{\kappa}_3 &= (3, 3, 3)^T - \frac{\vec{\kappa}_1 \cdot (3, 3, 3)^T}{\vec{\kappa}_1 \cdot \vec{\kappa}_1} \vec{\kappa}_1 - \frac{\vec{\kappa}_2 \cdot (3, 3, 3)^T}{\vec{\kappa}_2 \cdot \vec{\kappa}_2} \vec{\kappa}_2 \\ &= (3, 3, 3)^T - \frac{9}{7} (1, 2, 3)^T - \frac{10}{19} (33/14, 12/7, -27/14)^T \\ &= (9/19, -9/19, 3/19)^T \end{aligned}$$

Our orthogonal basis is

$$\left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 34/14 \\ 12/7 \\ -27/14 \end{pmatrix}, \begin{pmatrix} 9/19 \\ -9/19 \\ 3/19 \end{pmatrix} \right\rangle.$$

These vectors have not been normalized.

8. Find an orthonormal basis for this subspace of  $\mathbb{R}^4$ :

$$\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : x - y - z + w = 0 \text{ and } x + z = 0 \right\}.$$

### Solution

Let's look at the system

$$\begin{aligned}x - y - z + w &= 0, \\x + z &= 0.\end{aligned}$$

By using an augmented matrix and reducing to RREF, the solutions are given by

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} w; \quad z, w \in \mathbb{R}.$$

A basis for the solution space is

$$\left\langle \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle.$$

Now we use Gram-Schmidt...

$$\begin{aligned}\kappa_1 &= (-1, -2, 1, 0)^T \\ \kappa_2 &= (0, 1, 0, 1)^T - \frac{\kappa_1 \cdot (0, 1, 0, 1)^T}{\kappa_1 \cdot \kappa_1} \kappa_1 = (0, 1, 0, 1)^T + \frac{1}{3}(-1, -2, 1, 0)^T = (-1/3, 1/3, 1/3, 1)^T\end{aligned}$$

After normalizing each new vector, we get our orthonormal basis

$$\left\langle \begin{pmatrix} -\sqrt{6}/6 \\ -\sqrt{6}/3 \\ \sqrt{6}/6 \\ 0 \end{pmatrix}, \begin{pmatrix} -\sqrt{3}/6 \\ \sqrt{3}/6 \\ \sqrt{3}/6 \\ \sqrt{3}/2 \end{pmatrix} \right\rangle.$$

9. What happens if we apply the Gram-Schmidt process to a finite set that is not linearly independent?

### Solution

If the original set is dependent, the Gram-Schmidt process will eventually produce zero vectors. In fact, it will produce a zero vector whenever you apply the process to a vector that is a linear combination of the preceding vectors.

10. What happens if we apply the Gram-Schmidt process to a basis that is already orthogonal?

### Solution

The dot products will be zero, and the process will generate the original basis itself.