

Math 236 - Assignment 6

March 4, 2026

Name _____

Score _____

Show all work to receive full credit. Supply explanations when necessary. You may use technology to solve any linear systems. This assignment is due March 11.

1. Show that $f : \mathcal{M}_{2 \times 2} \rightarrow \mathbb{C}$ is a homomorphism.

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a + 2c) + (b - d)i$$

Solution

$$\begin{aligned} f\left(\alpha \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \beta \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) &= f\left(\begin{pmatrix} \alpha a_1 + \beta a_2 & \alpha b_1 + \beta b_2 \\ \alpha c_1 + \beta c_2 & \alpha d_1 + \beta d_2 \end{pmatrix}\right) \\ &= [(\alpha a_1 + \beta a_2) + 2(\alpha c_1 + \beta c_2)] + [(\alpha b_1 + \beta b_2) - (\alpha d_1 + \beta d_2)]i \\ &= \alpha[(a_1 + 2c_1) + (b_1 - d_1)i] + \beta[(a_2 + 2c_2) + (b_2 - d_2)i] \\ &= \alpha f\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\right) + \beta f\left(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) \end{aligned}$$

2. Let's make a slight change to the function defined in problem 1.

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a + 2) + (b - d)i$$

Show that f is not a homomorphism.

Solution

Notice that

$$f\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) = (0 + 2) + (0 - 0)i = 2.$$

Since f does not map the zero vector in $\mathcal{M}_{2 \times 2}$ to the zero vector in \mathbb{C} , it cannot be a homomorphism.

3. Let $n : \mathbb{R}^3 \rightarrow \mathbb{R}$ be defined by

$$n\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \sqrt{x^2 + y^2 + z^2}.$$

Give a simple example to show that n is not a homomorphism.

Solution

$$2\sqrt{3} = n\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) + n\left(\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}\right) \neq n\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}\right) = 0$$

4. Let $d : \mathbb{R}^3 \rightarrow \mathbb{R}$ be defined by

$$d\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + 2\hat{j} - 3\hat{k}).$$

Show that d is a homomorphism. (The centered dot denotes the dot product from Calculus III.)

Solution

First, expand the dot product to write

$$d\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = x + 2y - 3z.$$

Now,

$$\begin{aligned} d\left(\alpha \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \beta \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) &= d\left(\begin{pmatrix} \alpha x_1 + \beta x_2 \\ \alpha y_1 + \beta y_2 \\ \alpha z_1 + \beta z_2 \end{pmatrix}\right) \\ &= (\alpha x_1 + \beta x_2) + 2(\alpha y_1 + \beta y_2) - 3(\alpha z_1 + \beta z_2) = \alpha(x_1 + 2y_1 - 3z_1) + \beta(x_2 + 2y_2 - 3z_2) \\ &= \alpha d\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}\right) + \beta d\left(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right). \end{aligned}$$

5. Assume that $h : V \rightarrow W$ is a homomorphism. The *null space* of h is

$$\mathcal{N}(h) = \{\vec{v} \in V : h(\vec{v}) = \vec{0}_W\}.$$

Show that the null space is a subspace of V .

Solution

Let α and β be arbitrary scalars, and let \vec{x} and \vec{y} be arbitrary vectors in $\mathcal{N}(h)$. It follows that $\alpha h(\vec{x}) + \beta h(\vec{y}) = \vec{0}_W$ because \vec{x} and \vec{y} are in the null space of h . Furthermore, since h is a homomorphism,

$$\vec{0}_W = \alpha h(\vec{x}) + \beta h(\vec{y}) = h(\alpha\vec{x} + \beta\vec{y}).$$

Therefore, $\alpha\vec{x} + \beta\vec{y} \in \mathcal{N}(h)$.

6. Assume that $h : V \rightarrow W$ is a homomorphism. The *range* of h is

$$\mathcal{R}(h) = \{\vec{w} \in W : \vec{w} = h(\vec{v}) \text{ for some } \vec{v} \in V\}.$$

Show that the range is a subspace of W .

Solution

Let α and β be arbitrary scalars, and let \vec{x} and \vec{y} be arbitrary vectors in $\mathcal{R}(h)$. Because \vec{x} and \vec{y} are in the range space of h , there are vectors $\vec{u}, \vec{v} \in V$ such that

$$h(\vec{u}) = \vec{x} \quad \text{and} \quad h(\vec{v}) = \vec{y}.$$

Now since h is a homomorphism, it follows that

$$\alpha\vec{x} + \beta\vec{y} = \alpha h(\vec{u}) + \beta h(\vec{v}) = h(\alpha\vec{u} + \beta\vec{v}).$$

Finally, since V is a vector space, $\alpha\vec{u} + \beta\vec{v} \in V$, and we see that $\alpha\vec{x} + \beta\vec{y}$ is the image under h of a vector in V . That is, $\alpha\vec{x} + \beta\vec{y} \in \mathcal{R}(h)$.

7. Suppose $h : V \rightarrow V$ is a homomorphism and that $B = \langle \vec{\beta}_1, \vec{\beta}_2, \dots, \vec{\beta}_n \rangle$ is a basis for V . Prove the statement: If $h(\vec{\beta}_i) = \vec{0}$ for each basis vector, then h is the zero map.

Solution

Let \vec{v} be an arbitrary vector in V . Since B is a basis, there exist constants c_1, c_2, \dots, c_n such that

$$\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \dots + c_n\vec{\beta}_n.$$

Now, since h is a homomorphism and h maps each basis vector to $\vec{0}$, we have,

$$h(\vec{v}) = h(c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \dots + c_n\vec{\beta}_n) = c_1h(\vec{\beta}_1) + c_2h(\vec{\beta}_2) + \dots + c_nh(\vec{\beta}_n) = c_1\vec{0} + c_2\vec{0} + \dots + c_n\vec{0} = \vec{0}.$$

In summary, we've found that $h(\vec{v}) = \vec{0}$ for any vector in $\vec{v} \in V$.

8. For the map $h : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ given by

$$h\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2x + y \\ x - z \end{pmatrix},$$

find the range space, rank, null space, and nullity.

Solution

Range space...

$$\mathcal{R}(h) = \left\{ \begin{pmatrix} 2x + y \\ x - z \end{pmatrix} : x, y, z \in \mathbb{R} \right\} = \left\{ x \begin{pmatrix} 2 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ -1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

$$\begin{aligned}
&= \text{span}\left(\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\}\right) \\
&= \text{span}\left(\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}\right)
\end{aligned}$$

Any basis for \mathbb{R}^2 is a basis for $\mathcal{R}(h)$.

Rank...

$$\text{rank}(h) = \dim(\mathcal{R}(h)) = 2$$

Null space...

$$\begin{aligned}
\mathcal{N}(h) &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 2x + y = 0 \text{ and } x - z = 0 \right\} = \left\{ \begin{pmatrix} x \\ -2x \\ x \end{pmatrix} : x \in \mathbb{R} \right\} \\
&= \text{span}\left(\left\{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right\}\right)
\end{aligned}$$

Therefore, a basis for $\mathcal{N}(h)$ is

$$B_{\mathcal{N}} = \left\langle \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\rangle.$$

Nullity...

$$\dim(\mathcal{N}(h)) = 1$$

9. Define $F, G : \mathcal{P}_2 \rightarrow \mathbb{R}^2$ by

$$F(ax^2 + bx + c) = \begin{pmatrix} a + c \\ b \end{pmatrix} \quad \text{and} \quad G(ax^2 + bx + c) = \begin{pmatrix} 2b \\ -3c \end{pmatrix}.$$

Do not bother showing that F and G are both homomorphisms, but show that the linear combination $4F - 2G$ is a homomorphism from \mathcal{P}_2 into \mathbb{R}^2 .

Solution

The specific rules that actually define the homomorphisms are irrelevant.

Using that F and G are both homomorphisms, we have

$$\begin{aligned}
(4F - 2G)(\alpha x + \beta y) &= 4F(\alpha x + \beta y) - 2G(\alpha x + \beta y) \\
&= 4\alpha F(x) + 4\beta F(y) - 2\alpha G(x) - 2\beta G(y) \\
&= 4\alpha F(x) - 2\alpha G(x) + 4\beta F(y) - 2\beta G(y) \\
&= \alpha(4F - 2G)(x) + \beta(4F - 2G)(y).
\end{aligned}$$