MTH 236 Assignment #10 Key

1.) "Is similar to" is an equivalence relation.

1. $A \sim A$: $A = IAI^{-1}$ ✓ Similarity is reflexive.

2. Suppose $A \sim B$. Then $A = PBP^{-1}$ for some nonsing. matrix P.

   It follows that

   $B = P^{-1}AP$. Let $Q = P^{-1}$, then we have a nonsing. matrix with $B = QAQ^{-1}$.

   Therefore $B \sim A$. ✓

   Similarity is symmetric.

3. Suppose $A \sim B$ and $B \sim C$.

   Then there are matrices P & Q with $A = PBP^{-1}$ and $B = QCQ^{-1}$.

   It follows that $A = PQCQ^{-1}P^{-1}$

   $= (PQ)C(PQ)^{-1}$

   $= MCM^{-1}$, where $M = PQ$.

   Therefore $A \sim C$. ✓

   Similarity is transitive.

2) Suppose $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is diagonalizable by

$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where P is nonsingular.

Then
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}$$

or

$$\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a\delta_1 & b\delta_2 \\ c\delta_1 & d\delta_2 \end{pmatrix}$$

Case 1: $\delta_1 \neq 0$, $\delta_2 \neq 0$.

Then $c = d = 0$ and this contradicts P is nonsingular.

Case 2: $\delta_1 \neq 0$, $\delta_2 = 0$

Then $c = 0$ and $\therefore$ $a = 0$. This contradicts P is nonsing.

Case 3: $\delta_1 = 0$, $\delta_2 \neq 0$

Then $d = 0$ and $\therefore$ $b = 0$. This contradicts P is nonsing.

Case 4: $\delta_1 = \delta_2 = 0$.

Then $c = 0$ and $d = 0$. This contradicts P is nonsing.

There are no possibilities left. We cannot have

$$\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a\delta_1 & b\delta_2 \\ c\delta_1 & d\delta_2 \end{pmatrix}$$ with nonsing P.

3) $A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -1 & 6 \\ 2 & 1-\lambda & 6 \\ 2 & -1 & 8-\lambda \end{vmatrix}$$

$$= (4-\lambda)\left[(1-\lambda)(8-\lambda)+6\right] + 2(8-\lambda) - 12 + 6\left[-2-2(1-\lambda)\right]$$

$$= (4-\lambda)(1-\lambda)(8-\lambda) + 24 - 6\lambda + 16 - 2\lambda - 12 - 12 - 12 + 12\lambda$$

$$= 32 - 36\lambda + 4\lambda^2 - 8\lambda + 9\lambda^2 - \lambda^3 + 4 + 4\lambda$$

$$= 36 - 40\lambda + 13\lambda^2 - \lambda^3$$

$$\boxed{\text{CHAR POLY} = p(\lambda) = 36 - 40\lambda + 13\lambda^2 - \lambda^3}$$

$p(2) = 36 - 80 + 52 - 8 = 0$

$\Rightarrow \lambda = 2$ IS AN EIGENVALUE.

$$A - 2I = \begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & -1/2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

CORRESPONDING EIGENVECTORS ARE $\begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}$ AND $\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

$$\boxed{\text{EIGENSPACE HAS BASIS } \left\langle \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\rangle}$$

4) Char poly of A is

$$p(\lambda) = (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda)$$

$$\boxed{p(\lambda) = (5-\lambda)^2(3-\lambda)(1-\lambda)}$$

5) $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has only $\lambda = 1$ (mult 2)

6) Suppose A is a square matrix for which $A^2 = 0$.

Let λ be any eigenvalue of A

Then $A\vec{x} = \lambda\vec{x}$ for some $\vec{x} = \vec{0}$.

It follows that $A(A\vec{x}) = A^2\vec{x} = A\lambda\vec{x}$

$$= \lambda A\vec{x} = \lambda^2\vec{x} = 0$$

Since $\vec{x} \neq \vec{0}$, we must have $\lambda = 0$.

7.) $$\begin{vmatrix} 1-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) = 0$$

$$\Rightarrow \lambda = 1,\ \lambda = 2$$

$\underline{\lambda = 1}$

$$\begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \vec{X}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\underline{\lambda = 2}$

$$\begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \Rightarrow \vec{X}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\underbrace{A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix},\quad P = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix},\quad D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}}_{A = PDP^{-1}},\quad P^{-1} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

8.)

$$\begin{vmatrix} 6-\lambda & 10 \\ -2 & -3-\lambda \end{vmatrix} = (\lambda-6)(\lambda+3)+20$$

$$= \lambda^2-3\lambda+2 = (\lambda-2)(\lambda-1) = 0$$

$$\lambda = 1,\ \lambda = 2$$

$\underline{\lambda = 1}$

$$\begin{pmatrix} 5 & 10 \\ -2 & -4 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \Rightarrow \vec{X}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$\lambda = 2$

$$\begin{pmatrix} 4 & 10 \\ -2 & -5 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 5/2 \\ 0 & 0 \end{pmatrix} \Rightarrow \vec{X}_2 = \begin{pmatrix} -5/2 \\ 1 \end{pmatrix}$$

OR LET'S USE $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$

$$\underbrace{A = \begin{pmatrix} 6 & 10 \\ -2 & -3 \end{pmatrix},\quad P = \begin{pmatrix} -2 & 5 \\ 1 & -2 \end{pmatrix},\quad D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},\quad P^{-1} = -1\begin{pmatrix} -2 & -5 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}}_{A = PDP^{-1}}$$

9.) $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(-5-\lambda)(1-\lambda) + 9\right] - 3\left[-3(1-\lambda) + 9\right] + 3\left[-9 + 3(5+\lambda)\right]$$

$$= (1-\lambda)(\lambda^2 + 4\lambda + 4) - 3(6 + 3\lambda) + 3(6 + 3\lambda)$$

$$= (1-\lambda)(\lambda+2)^2 = 0 \Rightarrow \lambda = 1, \ \lambda = -2 \ (\text{mult } 2)$$

$\lambda = 1 \ldots$

$$A - I = \begin{pmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \vec{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$\lambda = -2 \ldots$

$$A + 2I = \begin{pmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \vec{x}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{x}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

NEXT PAGE →

$$P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$ From Sage.

$$PDP^{-1} = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix} = A$$ ✓