Math 236 - Final Exam

May 14, 2025

Name Key

Score

Show all work to receive full credit. Supply explanations when necessary. Unless otherwise indicated, you may use your calculator to obtain any RREF.

1. (10 points) Let $A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 3 & 1 \\ 2 & 15 & 4 \end{pmatrix}$. Do not use your calculator for this problem.

(a) Find a basis for the row space of A.

$-R_2 \leftrightarrow R_1$ $\begin{pmatrix} 1 & -3 & -1 \\ 2 & 1 & 0 \\ 2 & 15 & 4 \end{pmatrix}$

$R_2 = -2R_1 + R_2$, $R_3 = -2R_1 + R_3$ $\begin{pmatrix} 1 & -3 & -1 \\ 0 & 7 & 2 \\ 0 & 21 & 6 \end{pmatrix}$

$R_3 = -R_2 + R_3$, $R_2 = \frac{1}{7}R_2$ $\begin{pmatrix} 1 & -3 & -1 \\ 0 & 1 & 2/7 \\ 0 & 0 & 0 \end{pmatrix}$

$R_1 = 3R_2 + R_1$ $\begin{pmatrix} 1 & 0 & -1/7 \\ 0 & 1 & 2/7 \\ 0 & 0 & 0 \end{pmatrix}$

ROW SPACE BASIS $= \left\langle (1 \;\; 0 \;\; -1/7),\ (0 \;\; 1 \;\; 2/7) \right\rangle$

(b) Find a basis for the column space of A.

FROM RREF ABOVE, ORIGINAL COLUMNS 1 & 2 AND LIN. INDEP.

AND $\text{COL } 3 = -\frac{1}{7} \text{COL } 1 + \frac{2}{7} \text{COL } 2$

BASIS FOR COLUMN SPACE $= \left\langle \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 15 \end{pmatrix} \right\rangle$

(c) What is the rank of A. How do you know?

RANK OF A = DIM OF ROW SPACE

= DIM OF COLUMN SPACE

$= \boxed{2}$

2. (8 points) Find a basis for the subspace of $\mathcal{M}_{2\times 2}$ that is defined by

$$W = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + 2b - d = 0 \text{ and } 3b + d = 0\}.$$

What is the dimension of W?

$$\begin{aligned} a + 2b - d &= 0 \\ 3b + d &= 0 \\ \hline a + 5b &= 0 \end{aligned}$$

$a = -5b$

$d = -3b$

C IS ARBITRARY

$$W = \left\{ \begin{pmatrix} -5b & b \\ c & -3b \end{pmatrix} : b, c \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} -5 & 1 \\ 0 & -3 \end{pmatrix} b + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} c : b, c \in \mathbb{R} \right\}$$

BASIS FOR W =

$$\left\langle \begin{pmatrix} -5 & 1 \\ 0 & -3 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle$$

$\boxed{\dim(W) = 2}$

3. (8 points) Verify that $f : \mathcal{P}_2 \to \mathbb{R}^3$ is one-to-one and onto.

$$f(ax^2 + bx + c) = \begin{pmatrix} a + b \\ b \\ a + c \end{pmatrix}$$

1-1:

$f(ax^2+bx+c) = f(\alpha x^2 + \beta x + \gamma)$

$$\Rightarrow \begin{pmatrix} a+b \\ b \\ a+c \end{pmatrix} = \begin{pmatrix} \alpha+\beta \\ \beta \\ \alpha+\gamma \end{pmatrix}$$

$\Rightarrow$ ① $b = \beta$,

$a + b = \alpha + \beta$

$a + \beta = \alpha + \beta$

② $a = \alpha$,

$a + c = \alpha + \gamma$

$\alpha + c = \alpha + \gamma$

③ $c = \gamma$.

$$\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \checkmark$$

ONTO: LET $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \in \mathbb{R}^3$

$$\begin{pmatrix} a+b \\ b \\ a+c \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \Rightarrow \begin{aligned} b &= \beta \\ a &= \alpha - \beta \\ c &= \gamma - \alpha + \beta \end{aligned}$$

NOW LET $p(x) = (\alpha - \beta)x^2 + \beta x + (\gamma - \alpha + \beta)$

IT FOLLOWS THAT

$$f(p(x)) = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \checkmark$$

4. (10 points) Consider the homomorphism $h : \mathcal{P}_3 \to \mathcal{M}_{2\times 2}$ defined by

$$h(a + bx + cx^2 + dx^3) = \begin{pmatrix} d & a+2c \\ a+2c & a+2c \end{pmatrix}.$$

(a) Before you work any other parts of this problem, determine the sum of the rank of h and the nullity of h, and say how you know.

$$\text{rank}(h) + \text{nullity}(h) = \underbrace{\dim(\mathcal{P}_3)}_{\text{DIMENSION OF DOMAIN}} = \boxed{4}$$

(b) Find a basis for the range space of h. Then state the rank of h.

$$\text{RANGE} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} d + \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}(a+2c) : d, a, c \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} d + \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} e : d, e \in \mathbb{R} \right\}$$

$$\boxed{\text{BASIS IS } \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle}$$

$$\boxed{\text{rank}(h) = 2}$$

(c) Find a basis for the null space of h. Then state the nullity of h.

$$\begin{pmatrix} d & a+2c \\ a+2c & a+2c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow d = 0, \quad a = -2c,$$

C & b ARBITRARY

$$\text{NULL SPACE} = \{ -2c + bx + cx^2 : c, b \in \mathbb{R} \}$$

$$= \{ (-2 + x^2)c + (x)b : c, b \in \mathbb{R} \}$$

$$\boxed{\text{BASIS IS } \langle x^2 - 2, x \rangle} \qquad \boxed{\text{nullity}(h) = 2}$$

5. (10 points) Consider the bases $B, D \subseteq \mathcal{P}_2$.

$x^2 + 2x + 1$

$$B = \langle x, 1, x^2 \rangle, \qquad D = \langle 1, x+1, (x+1)^2 \rangle$$

(a) Find the change-of-basis matrix with respect to B, D.

$$\mathrm{Rep}_{B,D}(id) = \left(\mathrm{Rep}_D(x) \;\middle|\; \mathrm{Rep}_D(1) \;\middle|\; \mathrm{Rep}_D(x^2) \right)_{B,D}$$

$$= \boxed{\begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}_{B,D}}$$

Easy to mentally compute and confirm columns.

(b) Let $p(x) = 8x^2 + 5x + 3$. Find $\mathrm{Rep}_B(p)$. Then use your change-of-basis matrix to find $\mathrm{Rep}_D(p)$.

$$\boxed{\mathrm{Rep}_B(p) = \begin{pmatrix} 5 \\ 3 \\ 8 \end{pmatrix}_B}$$

Easy to obtain mentally

$$\mathrm{Rep}_D(p) = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}_{B,D} \begin{pmatrix} 5 \\ 3 \\ 8 \end{pmatrix}_B = \boxed{\begin{pmatrix} 6 \\ -11 \\ 8 \end{pmatrix}_D}$$

(c) Is your change-of-basis matrix singular or nonsingular? How do you know?

Change-of-basis matrix is always $\boxed{\text{Nonsingular}}$ because the identity map is an isomorphism.

Also, $\det(\mathrm{Rep}_{B,D}(id)) = -1 \neq 0$

6. (7 points) What does it means for two square matrices to be *similar*? Show that matrix similarity is transitive.

THE SQUARE MATRICES A & B ARE SIMILAR IF THERE EXISTS A NONSING. MATRIX P WITH $A = PBP^{-1}$.

SUPPOSE $A \sim B$ AND $B \sim C$.

$A = PBP^{-1}$ AND $B = QCQ^{-1}$

$$\therefore \underbrace{A = P(QCQ^{-1})P^{-1} = (PQ)C(PQ)^{-1}}_{A \sim C}$$

7. (12 points) For any two matrices A and B in $\mathcal{M}_{2\times 2}$, define

$$(A, B) = a_{1,1}\, b_{1,1} + a_{1,2}\, b_{1,2} + a_{2,1}\, b_{2,1} + a_{2,2}\, b_{2,2}.$$

(a) Prove that $(\cdot, \cdot)$ is an inner product.

① $(A,B) = (B,A)$ OBVIOUSLY, BECAUSE REAL NUMBER MULT. IS COMMUTATIVE.

② $(\alpha A + \beta B, C)$

$= (\alpha a_{11} + \beta b_{11}) c_{11} + \cdots + (\alpha a_{22} + \beta b_{22}) c_{22}$

$= \alpha a_{11} c_{11} + \cdots + \alpha a_{22} c_{22} + \beta b_{11} c_{11} + \cdots + \beta b_{22} c_{22}$

$= \alpha (A, C) + \beta (B, C)$

③ $(A, A) = a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2$

$=$ SUM OF SQUARES ≥ 0

④ $a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 = 0$

IFF $a_{11} = a_{12} = a_{21} = a_{22} = 0$.

(b) Find two matrices in $\mathcal{M}_{2\times 2}$ with nonzero entries that are orthogonal with respect to this inner product.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}; \quad B = \begin{pmatrix} 4 & 3 \\ -2 & -1 \end{pmatrix} \qquad (A, B) = 4 + 6 - 6 - 4 = 0$$

(c) True or false? The standard basis on $\mathcal{M}_{2\times 2}$ is an orthonormal basis with respect to this inner product.

$$\underbrace{\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle}_{\text{EASY TO CHECK MUTUALLY ORTHOG. AND NORM 1.}}$$

TRUE

8. (20 points) Consider the matrix

$$M = \begin{pmatrix} -6 & 0 & 5 \\ 0 & 4 & 0 \\ -10 & 0 & 9 \end{pmatrix}.$$

(a) Find the eigenvalues and corresponding eigenvectors of M. (Helpful suggestion: Expand your determinant along the middle row or column.)

$$M - \lambda I = \begin{pmatrix} -6-\lambda & 0 & 5 \\ 0 & 4-\lambda & 0 \\ -10 & 0 & 9-\lambda \end{pmatrix}$$

$$\det(M-\lambda I) = 4-\lambda\left[(-6-\lambda)(9-\lambda)+50\right] = 4-\lambda(\lambda^2-3\lambda-4) = (4-\lambda)(\lambda-4)(\lambda+1)$$

EIGENVALUES ARE $\lambda=4, \lambda=4, \lambda=-1$

$\lambda=4\ldots$

$$\begin{pmatrix} -10 & 0 & 5 \\ 0 & 0 & 0 \\ -10 & 0 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$$

I'LL USE $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ AND $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$\lambda=-1\ldots$

$$\begin{pmatrix} -5 & 0 & 5 \\ 0 & 5 & 0 \\ -10 & 0 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

(b) What are the geometric multiplicities of the eigenvalues?

$\lambda=4 \to$ TWO LINEAR INDEP EIGENVECTORS: $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ AND $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

GEOMETRIC MULT OF $\lambda=4$ IS 2.

$\lambda=-1 \to$ ONE EIGENVECTOR: $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

GEO MULT OF $\lambda=-1$ IS 1

(c) Is M diagonalizable? If so, find matrices P and D so that $M = PDP^{-1}$, where D is a diagonal matrix.

YES...

$$\boxed{D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}}$$

$\det(A) = 2(-5) = -10$ (EXPAND ALONG LAST COLUMN)

9. (7 points) Use the matrix adjoint to find A^{-1}. $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 3 & 1 & 2 \end{pmatrix}$.

SIGNED COFACTOR MATRIX

$$= \begin{pmatrix} -2 & -4 & 5 \\ -4 & 2 & 5 \\ 0 & 0 & -5 \end{pmatrix}$$

ADJOINT IS TRANSPOSE OF

$$\boxed{A^{-1} = -\frac{1}{10}\begin{pmatrix} -2 & -4 & 0 \\ -4 & 2 & 0 \\ 5 & 5 & -5 \end{pmatrix} = \begin{pmatrix} 1/5 & 2/5 & 0 \\ 2/5 & -1/5 & 0 \\ -1/2 & -1/2 & 1/2 \end{pmatrix}}$$

10. (8 points) Use induction to prove that for any positive integer n,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

TRUE FOR $n=1$ BECAUSE $1^2 = \frac{1(2)(3)}{6}$.

ASSUME TRUE FOR $n=k$ SO THAT $1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$.

NOW LET'S LOOK AT $1^2 + 2^2 + \cdots + k^2 + (k+1)^2$.

BY OUR INDUCTION HYPOTHESIS, THIS IS $\frac{k(k+1)(2k+1)}{6} + (k+1)^2$.

$$\frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} = \frac{k+1}{6}\left[k(2k+1) + 6(k+1)\right] = \frac{k+1}{6}(2k^2 + 7k + 6)$$

$$= \frac{k+1}{6}(2k+3)(k+2)$$

$$= \frac{(k+1)(k+2)(2k+3)}{6} \checkmark$$