Math 236 - Assignment 8

April 2, 2025

Name ______________________

Score ______________________

Show all work to receive full credit. Supply explanations when necessary. This assignment is due April 9.

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1. Show that A is invertible for any θ and find A^{-1}.
$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

2. For an invertible matrix A and a positive integer k, prove that $(A^k)^{-1} = (A^{-1})^k$. (Technically, this proof requires a technique called induction. We will eventually talk about induction. For now, to complete the proof, simply write out what A^k and $(A^{-1})^k$ mean, and show that their product is I.)

3. Argue that the inverse of any permutation matrix is its transpose. (You need not give a formal proof, just a compelling argument.)

4. Suppose that A and B are $n \times n$ matrices. Prove that $AB = I$ if and only if $BA = I$. (Helpful hint: From the fact that $\text{rank}(AB) = \text{rank}(BA) = n$, it follows that $\text{rank}(A) = \text{rank}(B) = n$.)

5. Find the change of basis matrix for $B, D \subseteq \mathbb{R}^2$.
$$B = \left\langle \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\rangle, \qquad D = \left\langle \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\rangle$$

6. Find the change of basis matrix for $B, D \subseteq \mathcal{P}_2$.
$$B = \langle 1, x, x^2 \rangle, \qquad D = \langle x^2, 1, x \rangle$$

7. Find bases such that this matrix represents the identity map with respect to those bases. (There are infinitely many possibilities. One approach is to make up any basis for the codomain, and then compute the corresponding basis for the domain.)
$$\begin{pmatrix} 3 & 1 & 4 \\ 2 & -1 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

8. Perform the Gram-Schmidt process on this basis for $\mathbb{R}^3$:
$$\left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \right\rangle.$$

9. Find an orthonormal basis for this subspace of $\mathbb{R}^4$:
$$\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : x - y - z + w = 0 \text{ and } x + z = 0 \right\}.$$

Turn over.

10. What happens if we apply the Gram-Schmidt process to a finite set that is not linearly independent?

11. What happens if we apply the Gram-Schmidt process to a basis that is already orthogonal?