Math 233 - Test 2

March 13, 2025

Name _____

Score _____

Show all work to receive full credit. Supply explanations where necessary.

1. (7 points) Let $\vec{r}(t) = (1 + \cos t)\,\hat{\imath} + 3\sin t\,\hat{\jmath} + t^2\,\hat{k}$. Find $\hat{T}(t)$ and then compute $\hat{T}(0)$.

2. (7 points) Let $\vec{r}(t) = 6\sin 2t\,\hat{\imath} + 6\cos 2t\,\hat{\jmath} + 5t\,\hat{k}$. Find $\hat{N}(t)$.

3. (8 points) The acceleration vector of a moving particle is given by $\vec{a}(t) = t\,\hat{\imath} + 4\,\hat{\jmath} - 32\,\hat{k}$. Suppose we also know that $\vec{v}(0) = 3\,\hat{\imath} + \hat{\jmath} - \hat{k}$ and $\vec{r}(0) = 2\,\hat{\jmath} + 4\,\hat{k}$. Find $\vec{r}(6)$.

4. (10 points) Starting from $t = 0$, find the arc length parameter s for the given vector-valued function. After you have found s, solve for t in terms of s.

$$\vec{r}(t) = e^t \cos t\,\hat{\imath} + e^t \sin t\,\hat{\jmath} + e^t\,\hat{k}$$

5. (9 points) A projectile is fired from the ground with an initial speed of 500 m/sec at an angle of elevation of 45°. To get credit for this problem, you must set up and use the vector-valued function that gives the position of the projectile at time t. (Use $g = 9.8\,\text{m/sec}^2$.)

   (a) When and how far away will the projectile strike the ground?

   (b) How high overhead will the projectile be when it is 5 km downrange?

   (c) What is the maximum height reached by the projectile?

6. (3 points) An object is moving in such a way that its position at time t is given by $\vec{r}(t) = (1 + 3t)\,\hat{\imath} + (t - 2)\,\hat{\jmath} - 3t\,\hat{k}$. Write the acceleration vector in the form $\vec{a}(t) = a_T\hat{T}(t) + a_N\hat{N}(t)$. Explain your answer.

7. (3 points) The speedometer of your car reads a steady 45 mph. Could you be accelerating? Explain.

8. (10 points) On your formula card, you can find the curvature formula for a curve defined in the xy-plane by $y = f(x)$. Let's derive and use that formula.

   (a) First, suppose you are given the equation $y = f(x)$. Briefly argue that the vector-valued function $\vec{r}(t) = t\,\hat{\imath} + f(t)\,\hat{\jmath}$ has the same graph.

   (b) Show that $\|\vec{r}\,'(t)\| = \sqrt{1 + (f'(t))^2}$.

   (c) With $\vec{r}\,''(t) = f''(t)\,\hat{\jmath}$, compute $\kappa = \dfrac{\|\vec{r}\,' \times \vec{r}\,''\|}{\|\vec{r}\,'\|^3}$ to obtain the formula.

   (d) Use the formula to compute the curvature of the graph of $y = \sin 9x$ at the point where $x = \pi/2$.

9. (10 points) Each of these equations defines a surface in 3-space. Briefly describe each surface.

   (a) $z = y^2 - 1$

   (b) $z^2 + y^2 - 4x^2 = 4$

   (c) $y = 4x^2 - z^2$

   (d) $z^2 + x^2 + 1 = y^2$

   (e) $x^2 + 4y^2 + 9z^2 = 1$

10. (4 points) Give an example of an equation of a...

   (a) cone whose cross sections parallel to the xz-plane are circles.

   (b) paraboloid opening up the z-axis whose cross sections parallel to the xy-plane are ellipses.

11. (6 points) Let $f(x, y) = \sqrt{y - x^2 - 2}$.

   (a) What is the domain of f?

   (b) Sketch the level curve $f(x, y) = 0$.

   (c) Describe the graph of f.

12. (6 points) Let $g(x, y, z) = \dfrac{x - y + z}{2x + y - z}$.

(a) Evaluate $g(1, 0, -2)$.

(b) What is the domain of g?

(c) Describe the level surface $g(x, y, z) = 2$.

13. (10 points) Find each limit or show that it does not exist.

(a) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^2 - 2xy + y^2}{x - y}$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x - y}{x + y}$

14. (7 points) Consider the limit: $\lim_{(x,y)\to(1,-1)} \dfrac{xy+1}{x^2-y^2}$.

(a) If you were to evaluate the limit along several paths through $(1,-1)$, which path must you definitely avoid and why?

(b) Use the two-path test to show that the limit does not exist.