Math 233 - Test 2
March 13, 2025

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (7 points) Let $\vec{r}(t) = (1+\cos t)\,\hat{\imath} + 3\sin t\,\hat{\jmath} + t^2\,\hat{k}$. Find $\hat{T}(t)$ and then compute $\hat{T}(0)$.

$$\vec{r}\,'(t) = -\sin t\,\hat{\imath} + 3\cos t\,\hat{\jmath} + 2t\,\hat{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{(-\sin t)^2 + (3\cos t)^2 + (2t)^2} = \sqrt{\sin^2 t + 9\cos^2 t + 4t^2}$$

$$\boxed{\hat{T}(t) = \frac{-\sin t\,\hat{\imath} + 3\cos t\,\hat{\jmath} + 2t\,\hat{k}}{\sqrt{\sin^2 t + 9\cos^2 t + 4t^2}}}$$

$$\hat{T}(0) = \frac{3\hat{\jmath}}{\sqrt{9}} = \hat{\jmath} \qquad \boxed{\hat{T}(0) = \hat{\jmath}}$$

2. (7 points) Let $\vec{r}(t) = 6\sin 2t\,\hat{\imath} + 6\cos 2t\,\hat{\jmath} + 5t\,\hat{k}$. Find $\hat{N}(t)$.

$$\vec{r}\,'(t) = 12\cos 2t\,\hat{\imath} - 12\sin 2t\,\hat{\jmath} + 5\hat{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{144\cos^2 2t + 144\sin^2 2t + 25} = \sqrt{144+25} = 13$$

$$\hat{T}(t) = \frac{12\cos 2t\,\hat{\imath} - 12\sin 2t\,\hat{\jmath} + 5\hat{k}}{13}$$

$$\hat{T}'(t) = \frac{-24\sin 2t\,\hat{\imath} - 24\cos 2t\,\hat{\jmath}}{13}$$

$$\|\hat{T}'(t)\| = \frac{24}{13} \qquad \boxed{\hat{N}(t) = -\sin 2t\,\hat{\imath} - \cos 2t\,\hat{\jmath}}$$

3. (8 points) The acceleration vector of a moving particle is given by $\vec{a}(t) = t\,\hat{\imath} + 4\,\hat{\jmath} - 32\,\hat{k}$. Suppose we also know that $\vec{v}(0) = 3\,\hat{\imath} + \hat{\jmath} - \hat{k}$ and $\vec{r}(0) = 2\,\hat{\jmath} + 4\,\hat{k}$. Find $\vec{r}(6)$.

$$\vec{v}(t) = \left(\tfrac{1}{2}t^2 + c_1\right)\hat{\imath} + \left(4t + c_2\right)\hat{\jmath} + \left(-32t + c_3\right)\hat{k}$$

$$\vec{v}(0) = 3\hat{\imath} + \hat{\jmath} - \hat{k} \Rightarrow c_1 = 3,\ c_2 = 1,\ c_3 = -1$$

$$\vec{r}(t) = \left(\tfrac{1}{6}t^3 + 3t + d_1\right)\hat{\imath} + \left(2t^2 + t + d_2\right)\hat{\jmath} + \left(-16t^2 - t + d_3\right)\hat{k}$$

$$\vec{r}(0) = 2\hat{\jmath} + 4\hat{k} \Rightarrow d_1 = 0,\ d_2 = 2,\ d_3 = 4$$

$$\vec{r}(t) = \left(\tfrac{1}{6}t^3 + 3t\right)\hat{\imath} + \left(2t^2 + t + 2\right)\hat{\jmath} + \left(-16t^2 - t + 4\right)\hat{k}$$

$$\boxed{\vec{r}(6) = 54\hat{\imath} + 80\hat{\jmath} - 578\hat{k}}$$

4. (10 points) Starting from $t = 0$, find the arc length parameter s for the given vector-valued function. After you have found s, solve for t in terms of s.

$$\vec{r}(t) = e^t \cos t\,\hat{\imath} + e^t \sin t\,\hat{\jmath} + e^t\,\hat{k}$$

$$\vec{r}\,'(t) = \left(-e^t \sin t + e^t \cos t\right)\hat{\imath} + \left(e^t \cos t + e^t \sin t\right)\hat{\jmath} + e^t\,\hat{k}$$

$$\|\vec{r}\,'(t)\| = e^t\sqrt{(\cos t - \sin t)^2 + (\cos t + \sin t)^2 + 1^2}$$

$$= e^t\sqrt{\underbrace{2\cos^2 t + 2\sin^2 t}_{2} + 1} = e^t\sqrt{3}$$

$$s(t) = \int_0^t \sqrt{3}\,e^\tau\,d\tau = \sqrt{3}\,e^\tau\Big|_0^t = \sqrt{3}\,e^t - \sqrt{3}$$

$$\boxed{s(t) = \sqrt{3}\,e^t - \sqrt{3} \quad \text{AND} \quad t = \ln\left(\frac{s + \sqrt{3}}{\sqrt{3}}\right)}$$

$\vec{r}(t) = 500\cos 45^\circ\, t\ \hat{\imath} + \left(-4.9t^2 + 500\sin 45^\circ t\right)\hat{\jmath}$

5. (9 points) A projectile is fired from the ground with an initial speed of 500 m/sec at an angle of elevation of 45°. To get credit for this problem, you must set up and use the vector-valued function that gives the position of the projectile at time t. (Use $g \doteq 9.8\,\text{m/sec}^2$.) $\vec{r}(t) = 250\sqrt{2}\,t\ \hat{\imath} + \left(-4.9t^2 + 250\sqrt{2}\,t\right)\hat{\jmath}$

   (a) When and how far away will the projectile strike the ground?

$$-4.9t^2 + 250\sqrt{2}\,t = 0 \Rightarrow t = \frac{250\sqrt{2}}{4.9} \approx \boxed{72.15 \text{ sec}}$$

$$250\sqrt{2}\left(\frac{250\sqrt{2}}{4.9}\right) \approx \boxed{25510.20 \text{ m}}$$

   (b) How high overhead will the projectile be when it is 5 km downrange?

$$250\sqrt{2}\,t = 5000 \Rightarrow t = \frac{20}{\sqrt{2}} \approx 14.14 \text{ sec}$$

$$-4.9\left(\frac{20}{\sqrt{2}}\right)^2 + 250\sqrt{2}\left(\frac{20}{\sqrt{2}}\right) = \boxed{4020.00 \text{ m}}$$

   (c) What is the maximum height reached by the projectile?

$$-9.8t + 250\sqrt{2} = 0 \Rightarrow t = \frac{250\sqrt{2}}{9.8} \approx 36.08 \text{ sec}$$

$$-4.9\left(\frac{250\sqrt{2}}{9.8}\right)^2 + 250\sqrt{2}\left(\frac{250\sqrt{2}}{9.8}\right) \approx \boxed{6377.55 \text{ m}}$$

6. (3 points) An object is moving in such a way that its position at time t is given by $\vec{r}(t) = (1+3t)\,\hat{\imath} + (t-2)\,\hat{\jmath} - 3t\,\hat{k}$. Write the acceleration vector in the form $\vec{a}(t) = a^T\hat{T}(t) + a_N\hat{N}(t)$. Explain your answer.

$\vec{r}(t)$ IS A LINEAR FUNCTION. MOTION IS ALONG A LINE WITH $\vec{v}(t)$ = CONSTANT. $\hat{N}(t)$ DOES NOT EXIST, AND $\hat{T}(t)$ IS CONSTANT WITH $a_T = 0$.

$$\boxed{\vec{a}(t) = 0\,\hat{T}(t)}$$

7. (3 points) The speedometer of your car reads a steady 45 mph. Could you be accelerating? Explain.

YES, EVEN THOUGH YOUR SPEED IS NOT CHANGING $(a_T = 0)$, YOU MAY BE CHANGING DIRECTION $(a_N \neq 0)$.

8. (10 points) On your formula card, you can find the curvature formula for a curve defined in the xy-plane by $y = f(x)$. Let's derive and use that formula.

(a) First, suppose you are given the equation $y = f(x)$. Briefly argue that the vector-valued function $\vec{r}(t) = t\,\hat{\imath} + f(t)\,\hat{\jmath}$ has the same graph.

ELIMINATE THE PARAMETER t:

$$\begin{aligned} x &= t \\ y &= f(t) \end{aligned} \Rightarrow y = f(x) \quad \checkmark$$

(b) Show that $\|\vec{r}\,'(t)\| = \sqrt{1 + (f'(t))^2}$.

$$\vec{r}\,'(t) = \hat{\imath} + f'(t)\,\hat{\jmath}$$

$$\|\vec{r}\,'(t)\| = \sqrt{(1)^2 + (f'(t))^2} = \sqrt{1 + (f'(t))^2} \quad \checkmark$$

(c) With $\vec{r}\,''(t) = f''(t)\,\hat{\jmath}$, compute $\kappa = \dfrac{\|\vec{r}\,' \times \vec{r}\,''\|}{\|\vec{r}\,'\|^3}$ to obtain the formula.

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & f'(t) & 0 \\ 0 & f''(t) & 0 \end{vmatrix} = f''(t)\,\hat{k}$$

$$\therefore \|\vec{r}\,'(t) \times \vec{r}\,''(t)\| = |f''(t)|$$

AND $$\kappa = \frac{\|\vec{r}\,' \times \vec{r}\,''\|}{\|\vec{r}\,'\|^3} = \frac{|f''(t)|}{\left[1 + (f'(t))^2\right]^{3/2}} \quad \checkmark$$

(d) Use the formula to compute the curvature of the graph of $y = \sin 9x$ at the point where $x = \pi/2$.

$$\begin{aligned} x &= t \\ y &= f(t) \\ &\Downarrow \\ y &= f(x) \end{aligned}$$

$$f(x) = \sin 9x$$

$$f'(x) = 9\cos 9x$$

$$f''(x) = -81 \sin 9x$$

AT $x = \frac{\pi}{2}$, $$\kappa = \frac{\left|-81 \sin \frac{9\pi}{2}\right|}{\left[1 + \left(9\cos\frac{9\pi}{2}\right)^2\right]^{3/2}} = \frac{81}{1} = \boxed{81}$$

9. (10 points) Each of these equations defines a surface in 3-space. Briefly describe each surface.

(a) $z = y^2 - 1$ $\boxed{\text{PARABOLIC CYLINDER.}}$ GENERATING CURVE IS A PARABOLA IN THE YZ-PLANE. RULINGS ARE PARALLEL TO X-AXIS

(b) $z^2 + y^2 - 4x^2 = 4$

FIX X: CIRCLES AT EVERY X

$z^2 + y^2 = 4 + 4x^2$

FIX Y & Z: HYPERBOLAS

$\boxed{\text{HYPERBOLOID ON ONE SHEET}}$ W/ CROSS SECTION ELLIPSES UP X-AXIS

(c) $y = 4x^2 - z^2$

FIX Y: HYPERBOLAS

FIX X & Z: PARABOLAS

$\boxed{\text{HYPERBOLIC PARABOLOID}}$

(d) $z^2 + x^2 + 1 = y^2$

FIX Y: CIRCLES FOR $|y| > 1$

$z^2 + x^2 = y^2 - 1$

FIX X & Z: HYPERBOLAS

$\boxed{\text{HYPERBOLOID OF TWO SHEETS}}$ W/ CROSS SECTION CIRCLES UP Y-AXIS

(e) $x^2 + 4y^2 + 9z^2 = 1$

$\boxed{\text{ELLIPSOID}}$ CENTERED AT $(0,0,0)$

10. (4 points) Give an example of an equation of a...

(a) cone whose cross sections parallel to the xz-plane are circles.

$$x^2 + z^2 = y^2$$

(b) paraboloid opening up the z-axis whose cross sections parallel to the xy-plane are ellipses.

$$z = x^2 + 2y^2$$

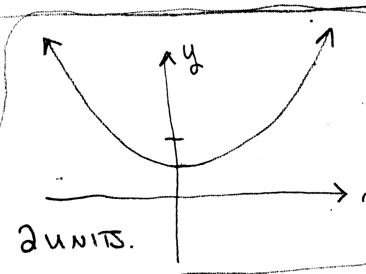
11. (6 points) Let $f(x, y) = \sqrt{y - x^2 - 2}$.

(a) What is the domain of f? $y - x^2 - 2 \geq 0 \Rightarrow y \geq x^2 + 2$

$\boxed{\text{DOMAIN} = \{(x,y) : y \geq x^2 + 2\}}$

(b) Sketch the level curve $f(x, y) = 0$.

$f(x,y) = 0 \Rightarrow y - x^2 - 2 = 0 \Rightarrow y = x^2 + 2$



$y = x^2$ PARABOLA SHIFTED UP 2 UNITS.

(c) Describe the graph of f.

$z^2 = y - x^2 - 2, \quad z \geq 0$

$y = x^2 + z^2 + 2$

THE GRAPH OF f IS THE $z \geq 0$-HALF OF A CIRCULAR PARABOLOID OPENING UP Y-AXIS. VERTEX $(0,2,0)$

12. (6 points) Let $g(x, y, z) = \dfrac{x - y + z}{2x + y - z}$.

(a) Evaluate $g(1, 0, -2)$.

$$g(1,0,-2) = \frac{1-0+(-2)}{2(1)+0-(-2)} = \boxed{\frac{-1}{4}}$$

(b) What is the domain of g?

$$\boxed{\text{DOMAIN} = \{ (x,y,z) : 2x+y-z \neq 0 \}}$$

(c) Describe the level surface $g(x, y, z) = 2$.

$$\frac{x-y+z}{2x+y-z} = 2 \Rightarrow x-y+z = 4x+2y-2z$$

$$\Rightarrow 3x+3y-3z = 0$$

LEVEL SURFACE IS A PLANE WITH $\vec{n} = 3\hat{i} + 3\hat{j} - 3\hat{k}$ PASSING THROUGH $(0,0,0)$

13. (10 points) Find each limit or show that it does not exist.

(a) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^2 - 2xy + y^2}{x - y}$ $\quad 0/0$

$$= \lim_{(x,y)\to(1,1)} \frac{(x-y)^2}{x-y} = \lim_{(x,y)\to(1,1)} (x-y) =$$

$$1-1 = \boxed{0}$$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x - y}{x + y}$ $\quad 0/0$

LET'S TRY SOME PATHS...

$x=0$: $\displaystyle\lim_{y\to 0} \frac{-y}{y} = \lim_{y\to 0} (-1) = -1$

$y=0$: $\displaystyle\lim_{x\to 0} \frac{x}{x} = \lim_{x\to 0} (1) = 1$

TWO DIFFERENT LIMITS ALONG TWO PATHS $\Rightarrow$ $\boxed{\text{LIMIT DNE}}$

14. (7 points) Consider the limit: $\lim_{(x,y)\to(1,-1)} \dfrac{xy+1}{x^2-y^2}$.

(a) If you were to evaluate the limit along several paths through $(1,-1)$, which path must you definitely avoid and why?

YOU MUST AVOID THE PATH WHERE $x^2 = y^2$. THAT IS, $y = -x$.

THIS PATH IS NOT IN THE DOMAIN OF $\dfrac{xy+1}{x^2-y^2}$.

(b) Use the two-path test to show that the limit does not exist.

$$y=-1: \quad \lim_{x\to 1} \frac{1-x}{x^2-1} = \lim_{x\to 1} \frac{1-x}{(x-1)(x+1)} = \lim_{x\to 1} \frac{-\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} = -\frac{1}{2}$$

$$x=1: \quad \lim_{y\to -1} \frac{y+1}{1-y^2} = \lim_{y\to -1} \frac{\cancel{y+1}}{(1-y)\cancel{(1+y)}} = \frac{1}{2}$$

TWO DIFF. LIMITS ALONG TWO PATHS

$\Downarrow$

LIMIT DNE.