Math 233 - Final Exam A

May 8, 2021

Name key

Score

Show all work to receive full credit. Each problem is worth 5 points. Place your final answer in the box provided. This test is due May 13.

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1. A baseball, hit 3 feet above the ground, leaves the bat at an angle of 45° and is caught by an outfielder at a height of 3 feet above the ground and 300 feet from home plate. What is the initial speed of the ball? (To receive full credit, you must write and use the vector-valued function $\vec{r}(t)$ that gives the position of the ball at time t. Also ignore air resistance and use $g \approx 32\,\text{ft}\,\text{s}^{-2}$.)

$$\vec{r}(t) = V_0 \cos 45^\circ t\ \hat{i} + \left(-16t^2 + V_0 \sin 45^\circ t + 3\right)\hat{j}$$

$$= \frac{V_0\sqrt{2}}{2} t\ \hat{i} + \left(-16t^2 + \frac{V_0\sqrt{2}}{2} t + 3\right)\hat{j}$$

$$\frac{V_0\sqrt{2}}{2} t = 300, \quad -16t^2 + \frac{V_0\sqrt{2}}{2} t + 3 = 3$$

$$\Rightarrow -16t^2 + 300 = 0 \Rightarrow t = \sqrt{\frac{300}{16}}$$

$$V_0 = \frac{600}{\sqrt{2}\,t} = \frac{600}{\sqrt{2}\sqrt{\frac{300}{16}}}$$

$$\boxed{V_0 = 40\sqrt{6}\ \text{FT/s} \approx 97.9796\ \text{FT/s}}$$

2. Find the limit or show that it does not exist: $\displaystyle\lim_{(x,y)\to(2,1)} \frac{x-y-1}{\sqrt{x-y}-1}$ $\frac{0}{0}$

$$\lim_{(x,y)\to(2,1)} \frac{x-y-1}{\sqrt{x-y}-1} \cdot \frac{\sqrt{x-y}+1}{\sqrt{x-y}+1}$$

$$= \lim_{(x,y)\to(2,1)} \frac{\cancel{(x-y-1)}(\sqrt{x-y}+1)}{\cancel{(x-y-1)}} = \boxed{2}$$

$$\boxed{2}$$

3. Find the limit or show that it does not exist: $\lim_{(x,y)\to(1,1)} \frac{x^2-y^2-2x+2y}{2x^2-4x+y+1}$

% MORE WORK

TRY SOME PATHS THROUGH (1,1) ...

$$y=1: \quad \lim_{x\to 1} \frac{x^2-2x+1}{2x^2-4x+2} = \lim_{x\to 1} \frac{(x-1)^2}{2(x-1)^2} = \frac{1}{2}$$

$$y=x: \quad \lim_{x\to 1} \frac{0}{2x^2-3x+1} = \lim_{x\to 1} 0 = 0$$

LIMIT DNE

$\boxed{\text{LIMIT DNE BY TWO-PATH TEST}}$

4. Find an equation of the plane tangent to the graph of $z = \tan^{-1}(y/x)$ at the point $\left(1, 1, \frac{\pi}{4}\right)$.

$$F(x,y,z) = \tan^{-1}\left(\frac{y}{x}\right) - z$$

$$\vec{\nabla} F(x,y,z) = \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2}\hat{i} + \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}\hat{j} - \hat{k}$$

$$\vec{n} = \vec{\nabla} F\left(1,1,\frac{\pi}{4}\right) = -\frac{1}{2}\hat{i} + \frac{1}{2}\hat{j} - \hat{k}$$

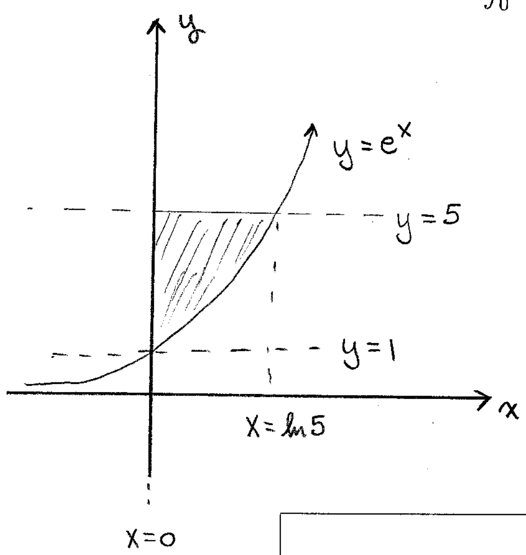
TAN PLANE IS $-\frac{1}{2}(x-1) + \frac{1}{2}(y-1) - \left(z - \frac{\pi}{4}\right) = 0$

OR

$$\boxed{-\frac{1}{2}x + \frac{1}{2}y - z = -\frac{\pi}{4}}$$

5. Reverse the order of integration and evaluate.

$$\int_0^{\ln 5} \int_{e^x}^{5} \frac{1}{\ln y}\, dy\, dx$$



$$\int_{y=1}^{y=5} \int_{x=0}^{x=\ln y} \frac{1}{\ln y}\, dx\, dy$$

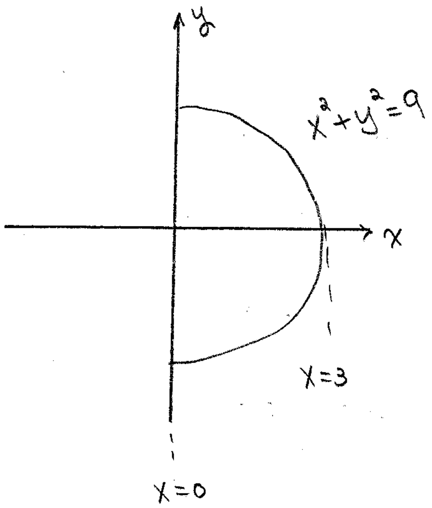
$$= \int_1^5 \frac{x}{\ln y}\Big|_0^{\ln y}\, dy$$

$$= \int_1^5 1\, dy = 5-1 = 4$$

$$\boxed{4}$$

6. Evaluate the iterated integral by first converting to cylindrical coordinates.

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{x^2+y^2}^{9} dz\, dy\, dx$$



$$\int_{-\pi/2}^{\pi/2} \int_0^3 \int_{r^2}^{9} r\, dz\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^3 rz\Big|_{r^2}^{9}\, dr\, d\theta$$

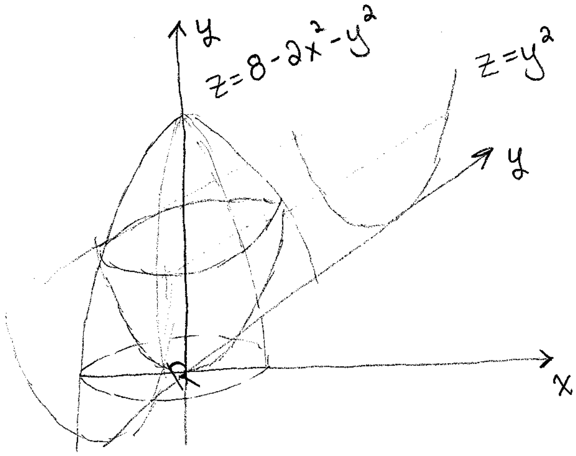
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^3 (9r - r^3)\, dr\, d\theta$$

$$= \pi\left[\frac{9}{2}r^2 - \frac{1}{4}r^4\right]_0^3$$

$$= \pi\left(\frac{81}{2} - \frac{81}{4}\right)$$

$$\boxed{\frac{81\pi}{4}}$$

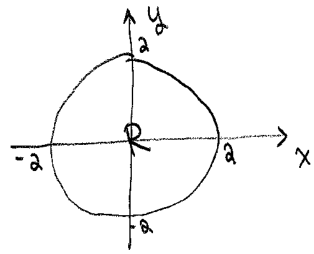
7. A solid in space is bounded above by the elliptic paraboloid $z = 8 - 2x^2 - y^2$ and below by the cylinder $z = y^2$. The density of the solid at the point (x, y, z) is given by $\rho(x, y, z) = z + x^2 + y^2$. Set up the iterated integral that gives the mass of the solid. DO NOT EVALUATE. (You may use whichever coordinate system you prefer.)



INTERSECTION ?

$8-2x^2-y^2 = y^2 \implies 8-2x^2-2y^2 = 0$

$x^2+y^2 = 4$



$$\boxed{\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}}\int_{y^2}^{8-2x^2-y^2} (z + x^2 + y^2)\, dz\, dy\, dx}$$

8. Let $\vec{F}(x, y, z) = y\hat{\imath} + x^2\hat{\jmath} - z\hat{k}$. Evaluate $\int_C \vec{F}(x, y, z) \cdot d\vec{r}$, where C is the line segment from $\underbrace{(0, 1, 1) \text{ to } (1, 3, 5)}$.

C :

$\vec{v} = \hat{\imath} + 2\hat{\jmath} + 4\hat{k}$

$x = t$

$y = 2t+1 \qquad 0 \le t \le 1$

$z = 4t+1$

$d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$

$= dt\,\hat{\imath} + 2\,dt\,\hat{\jmath} + 4\,dt\,\hat{k}$

$\vec{F} \cdot d\vec{r} = y\,dt + x^2(2)\,dt - z(4)\,dt$

$= (2t+1)\,dt + 2t^2\,dt - (16t+4)\,dt$

$= (2t^2 - 14t - 3)\,dt$

$\int_0^1 (2t^2 - 14t - 3)\,dt = \frac{2}{3} - \frac{14}{2} - 3 = -\frac{28}{3}$

$$\boxed{-\frac{28}{3}}$$

Math 233 - Final Exam B

May 13, 2021

Name key

Score

Show all work to receive full credit. Each problem is worth 5 points. Place your final answer in the box provided.

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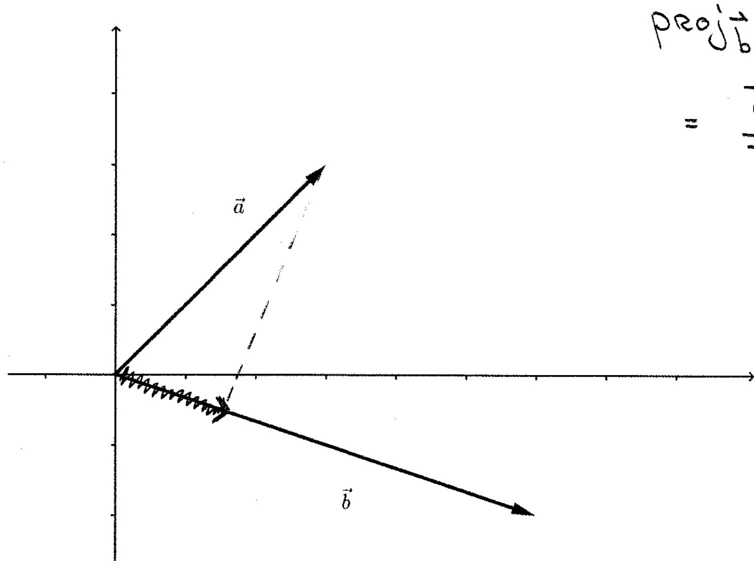
1. The vectors $\vec{a} = 3\hat{\imath} + 3\hat{\jmath}$ and $\vec{b} = 6\hat{\imath} - 2\hat{\jmath}$ are shown below. Sketch and compute $\text{proj}_{\vec{b}}\,\vec{a}$.



$\text{proj}_{\vec{b}}\,\vec{a}$

$= \dfrac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\vec{b} = \dfrac{18-6}{36+4}\vec{b}$

$= \dfrac{12}{40}\vec{b} = \dfrac{3}{10}\vec{b}$

$= \dfrac{9}{5}\hat{\imath} - \dfrac{3}{5}\hat{\jmath}$

$$\boxed{\text{proj}_{\vec{b}}\,\vec{a} = \frac{9}{5}\hat{\imath} - \frac{3}{5}\hat{\jmath}}$$

2. Given the points $A\,(5,3,1)$, $B\,(3,2,3)$, and $C\,(-4,-1,2)$, find a nonzero vector that is orthogonal to both $\vec{AB}$ and $\vec{AC}$.

$\vec{AB} = -2\hat{\imath} - \hat{\jmath} + 2\hat{k}$

$\vec{AC} = -9\hat{\imath} - 4\hat{\jmath} + \hat{k}$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & -1 & 2 \\ -9 & -4 & 1 \end{vmatrix} = \hat{\imath}(7) - \hat{\jmath}(16) + \hat{k}(-1)$$

$= 7\hat{\imath} - 16\hat{\jmath} - \hat{k}$

$$\boxed{\vec{AB} \times \vec{AC} = 7\hat{\imath} - 16\hat{\jmath} - \hat{k}}$$

3. The line ℓ has the symmetric equations $-x = \frac{y-1}{3} = \frac{z}{2}$. Find a set of parametric equations for the line through $(5, 2, -4)$ and parallel to ℓ.

$\vec{v} = -\hat{i} + 3\hat{j} + 2\hat{k}$

Point $(5, 2, -4)$

$$\boxed{x = 5 - t, \quad y = 2 + 3t, \quad z = -4 + 2t}$$

4. Find the measure of the acute angle between the planes given below. Give your final answer in degrees, rounded to the nearest tenth.

$$3x + 2y - 7z = 0$$

$$-4x + 4y + 2z = 13$$

$\vec{n}_1 = 3\hat{i} + 2\hat{j} - 7\hat{k}$

$\vec{n}_2 = -4\hat{i} + 4\hat{j} + 2\hat{k}$

$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \, \|\vec{n}_2\|} = \frac{-18}{\sqrt{62}\sqrt{36}}$

$\theta = \cos^{-1}\left(\frac{-18}{\sqrt{62}\sqrt{36}}\right) \approx 112.3957$

Not acute angle so subtract from $180°$

$$\boxed{67.6°}$$

$$\vec{r}\,'(t) = 3\hat{\imath} + 2\hat{\jmath} - \hat{k}$$

5. Find the length of the graph of $\vec{r}(t) = (3t-1)\hat{\imath} + (2t+7)\hat{\jmath} - (t-5)\hat{k}$ from the point where $t = 1$ to the point where $t = 3$.

$$\int_1^3 \|\vec{r}\,'(t)\|\,dt = \int_1^3 \sqrt{9+4+1}\,dt = \int_1^3 \sqrt{14}\,dt$$

$$= \boxed{2\sqrt{14}}$$

$$\boxed{2\sqrt{14}}$$

6. Suppose f is a function of the three variables x, y, and z. Choose two different fourth-order partial derivatives of f and state the conditions under which they will be equal to one another.

EQUAL IF

1. MIXED SAME WAY
2. CONTINUOUS

$f_{xxyz} = f_{xyzx}$ PROVIDED BOTH ARE CONTINUOUS ON AN OPEN REGION.

7. Use differentials to estimate the change in $f(x,y,z) = x^2 \ln(5yz+1)$ as (x,y,z) changes from $(2,1,1)$ to $(1.99, 1.02, 1.01)$. Round your final answer to four decimal places.

$$dz = 2x \ln(5yz+1)\,dx + \frac{5x^2z}{5yz+1}\,dy + \frac{5x^2y}{5yz+1}\,dz$$

$$\Delta z \approx (4 \ln 6)(-0.01) + \left(\frac{20}{6}\right)(0.02) + \left(\frac{20}{6}\right)(0.01)$$

$$\Delta z \approx 0.02833$$

$$\boxed{\Delta z \approx 0.0283}$$

8. Find the directional derivative of $g(x,y,z) = xye^z$ at $P\,(2,4,0)$ in the direction toward the point $Q\,(0,0,0)$.

$$\vec{\nabla} g(x,y,z) = ye^z\,\hat{i} + xe^z\,\hat{j} + xye^z\,\hat{k}$$

$$\vec{\nabla} g(2,4,0) = 4\hat{i} + 2\hat{j} + 8\hat{k}$$

$$\vec{PQ} = -2\hat{i} - 4\hat{j}$$

$$\|\vec{PQ}\| = \sqrt{20}$$

$$\vec{\nabla} g(2,4,0) \cdot \frac{\vec{PQ}}{\|\vec{PQ}\|} = \frac{-16}{\sqrt{20}}$$

$$\boxed{\frac{-16}{\sqrt{20}} = \frac{-8}{\sqrt{5}} \approx -3.5777}$$

9. Over a certain region in space the electrical potential V is given by

$$V(x, y, z) = 5x^2 - 3xy + xyz.$$

At the point $(3, 4, 5)$, in which direction does the potential decrease most rapidly?

$$\vec{\nabla} V(x,y,z) = (10x - 3y + yz)\hat{i} + (-3x + xz)\hat{j} + xy\,\hat{k}$$

$$\vec{\nabla} V(3,4,5) = 38\hat{i} + 6\hat{j} + 12\hat{k}$$

↑ OPPOSITE THIS DIRECTION

$$\boxed{-38\hat{i} - 6\hat{j} - 12\hat{k}}$$

10. Find the critical point(s) of $f(x, y) = x^2 + xy + 2y^2 + x - 3y + 10$. Then use the 2nd-partials test to classify the critical point(s).

$f_x(x,y) = 2x + y + 1 = 0$

$f_y(x,y) = x + 4y - 3 = 0$

$$\begin{aligned} 2x + y &= -1 \\ -2(x + 4y &= 3) \\ \hline -7y &= -7 \\ y &= 1 \end{aligned}$$

$x = 3 - 4(1)$
$= -1$

CRIT PT IS $(-1, 1)$

$d(x,y) = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 8 - 1 = 7 > 0$

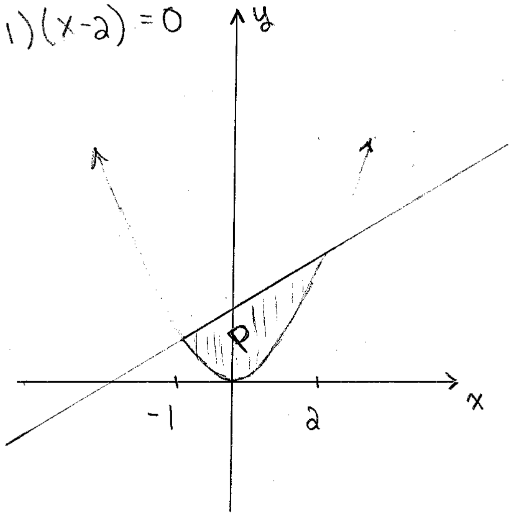
AND $f_{xx} = 2 > 0$

$\boxed{f(-1,1) = 8 \text{ IS A RELATIVE MIN}}$

11. Let P be the plane region between the graphs of $y = x^2$ and $y = x + 2$. Sketch the region P and then evaluate the double integral given below.

$x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$



$$\iint_P (x+2)\,dA$$

$$\int_{-1}^{2}\int_{x^2}^{x+2} (x+2)\,dy\,dx$$

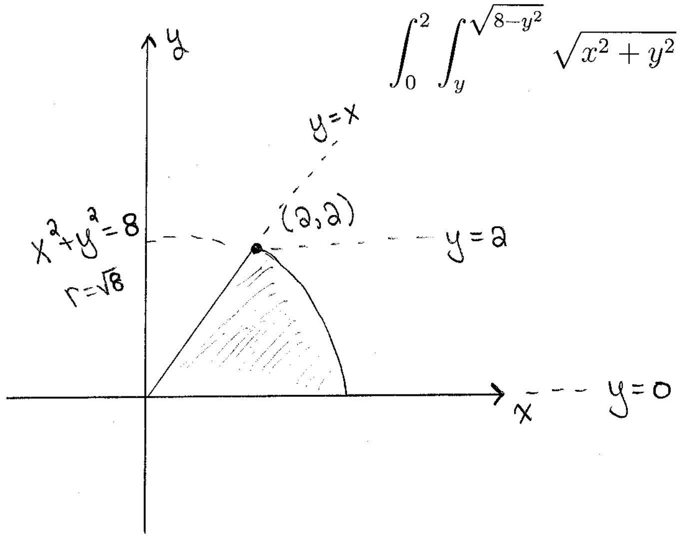
$$= \int_{-1}^{2} (xy + 2y)\Big|_{y=x^2}^{y=x+2} dx = \int_{-1}^{2} (x^2 + 2x + 2x + 4 - x^3 - 2x^2)\,dx$$

$$= \int_{-1}^{2} (-x^3 - x^2 + 4x + 4)\,dx = -\tfrac{1}{4}x^4 - \tfrac{1}{3}x^3 + 2x^2 + 4x\Big|_{-1}^{2} =$$

$$\frac{28}{3} - \left(-\frac{23}{12}\right)$$

$$\boxed{\frac{45}{4} = 11.25}$$

12. Evaluate the iterated integral by converting to polar coordinates.



$$\int_0^2 \int_y^{\sqrt{8-y^2}} \sqrt{x^2+y^2}\,dx\,dy$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{4}} \int_{r=0}^{r=\sqrt{8}} r^2\,dr\,d\theta$$

$$= \frac{\pi}{4}\int_0^{\sqrt{8}} r^2\,dr = \frac{\pi}{4}\,\frac{1}{3}r^3\Big|_0^{\sqrt{8}}$$

$$= \frac{2\pi\sqrt{8}}{3}$$

$$\boxed{\frac{2\pi\sqrt{8}}{3} = \frac{4\pi\sqrt{2}}{3}}$$