Math 233 - Quiz 3

February 25, 2021

Name key

Score ________

Show all work to receive full credit. Supply explanations when necessary. This quiz is due March 4.

1. (3 points) A projectile is launched from a bench above the ground. It is launched with an initial speed of 100 feet per second and at an angle of 30° above the horizontal. The projectile hits the ground after it has covered a horizontal distance of 276 ft. How high is the bench?

$$\vec{r}(t) = (100\cos 30^\circ)t\,\hat{i} + (-16t^2 + 100\sin 30^\circ t + y_0)\hat{j}$$

$$= 50\sqrt{3}\,t\,\hat{i} + (-16t^2 + 50t + y_0)\hat{j}$$

$50\sqrt{3}\,t = 276$

$t = \frac{276}{50\sqrt{3}} \approx 3.187\ s$

$$-16\left(\frac{276}{50\sqrt{3}}\right)^2 + 50\left(\frac{276}{50\sqrt{3}}\right) + y_0 = 0$$

$$\Rightarrow y_0 = \frac{101568}{625} - 92\sqrt{3}$$

≈ 3.16 FT

2. (2 points) Reparameterize the position vector below in terms of the arc length parameter.

$$\vec{r}(t) = (2 + 3t)\hat{\imath} - 5t\hat{\jmath} + (1 + t)\hat{k}, \qquad 0 \le t \le 2$$

$t_0 = 0$

$\vec{v}(t) = 3\hat{i} - 5\hat{j} + \hat{k}$

$\|\vec{v}(t)\| = \sqrt{9+25+1}$

$= \sqrt{35}$

$s(t) = \int_0^t \sqrt{35}\, d\tau = \sqrt{35}\,t$

$\frac{s}{\sqrt{35}} = t$

$$\vec{r}(s) = \left(2 + \frac{3s}{\sqrt{35}}\right)\hat{i} - \frac{5}{\sqrt{35}}s\,\hat{j} + \left(1 + \frac{s}{\sqrt{35}}\right)\hat{k}$$

$0 \le s \le 2\sqrt{35}$

Turn over.

3. (2 points) At what point does the graph of $y = x^2$ have the greatest curvature? Find a set of parametric equations for the graph. Then compute the curvature at the point of greatest curvature.

Using K Formula ...

$$f(x) = x^2, \quad K = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}} = \frac{2}{(1+4x^2)^{3/2}}$$

Same since $x = t$

Following directions...

$$\boxed{\begin{array}{l} x = t \\ y = t^2 \end{array} \Rightarrow \vec{r}(t) = t\hat{i} + t^2\hat{j}}$$

$$\vec{v}(t) = \hat{i} + 2t\hat{j}$$

$$\vec{a}(t) = 2\hat{j}$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix} = 2\hat{k}$$

$$K = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|^3} = \frac{2}{\left(\sqrt{1+4t^2}\right)^3}$$

$1 + 4t^2$ is clearly a min when $t = 0 \Rightarrow K$ is a max at $t = 0$ or $(x,y) = (0,0)$.

$$\boxed{K(t=0) = 2}$$

4. (3 points) Find the principal unit normal vector at $t = 0$.

$$\vec{r}(t) = \sqrt{2}\,t\hat{\imath} + e^t\hat{\jmath} + e^{-t}\hat{k}$$

$$\vec{v}(t) = \sqrt{2}\,\hat{i} + e^t\hat{j} - e^{-t}\hat{k}$$

$$\|\vec{v}(t)\| = \sqrt{2 + e^{2t} + e^{-2t}}$$

$$= \sqrt{(e^t + e^{-t})^2}$$

$$= e^t + e^{-t}$$

$$\hat{T}(t) = \frac{\sqrt{2}\,\hat{i} + e^t\hat{j} - e^{-t}\hat{k}}{e^t + e^{-t}}$$

$$\hat{T}'(t) = \frac{(e^t + e^{-t})\left[e^t\hat{j} + e^{-t}\hat{k}\right] - \left[\sqrt{2}\,\hat{i} + e^t\hat{j} - e^{-t}\hat{k}\right](e^t - e^{-t})}{(e^t + e^{-t})^2}$$

$$\hat{T}'(0) = \frac{2(\hat{j} + \hat{k})}{4} = \frac{1}{2}(\hat{j} + \hat{k})$$

$$\|\hat{T}'(0)\| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}}$$

$$\boxed{\hat{N}(0) = \frac{1}{\sqrt{2}}(\hat{j} + \hat{k})}$$