Math 233 - Test 1

September 10, 2026

Name key

Score ________

Show all work to receive full credit. Supply explanations where necessary.

1. (9 points) In the following problems, the vectors $\vec{u}$ and $\vec{w}$ are 2D vectors in the xy-plane.

   (a) The vector $\vec{w}$ has magnitude 7 and makes a 150° angle with the positive x-axis. Find the exact component form of $\vec{w}$.

   $$\vec{w} = 7\cos 150^\circ\,\hat{\imath} + 7\sin 150^\circ\,\hat{\jmath}$$

   $$\boxed{\vec{w} = -\frac{7\sqrt{3}}{2}\hat{\imath} + \frac{7}{2}\hat{\jmath}}$$

   (b) The vector $\vec{u}+\vec{w}$ has component form $\sqrt{3}\,\hat{\imath} - 4\,\hat{\jmath}$. Find the exact component form of $\vec{u}$.

   $$\vec{u} = (\sqrt{3}\,\hat{\imath} - 4\hat{\jmath}) - \left(-\frac{7\sqrt{3}}{2}\hat{\imath} + \frac{7}{2}\hat{\jmath}\right)$$

   $$\boxed{\vec{u} = \frac{9\sqrt{3}}{2}\hat{\imath} - \frac{15}{2}\hat{\jmath}}$$

   (c) What angle does $\vec{u}$ make with the positive x-axis?

   $\vec{u}$ IS IN QUAD IV AND $\tan\theta = \dfrac{-15/2}{9\sqrt{3}/2} = -\dfrac{5}{3\sqrt{3}}$

   $$\theta = \boxed{\tan^{-1}\left(\frac{-5}{3\sqrt{3}}\right) \approx -43.9^\circ}$$

2. (5 points) Find a vector of magnitude 3 that is parallel to the line with symmetric equations

   $$\underbrace{\frac{x-4}{3} = \frac{6-y}{5} = z.}$$

   $$\frac{x-4}{3} = \frac{y-6}{-5} = \frac{z-0}{1}$$

   $$\vec{v} = 3\hat{\imath} - 5\hat{\jmath} + \hat{k}$$

   $$\|\vec{v}\| = \sqrt{9+25+1} = \sqrt{35}$$

   $$\frac{3\vec{v}}{\|\vec{v}\|} = \boxed{\frac{1}{\sqrt{35}}\left(9\hat{\imath} - 15\hat{\jmath} + 3\hat{k}\right)}$$

   $$\approx 1.52\hat{\imath} - 2.54\hat{\jmath} + 0.51\hat{k}$$

3. (8 points) Find the acute angle between the planes described by the equations below. Write your final answer in degrees, rounded to the nearest hundredth.

$$3x + 4y - 9z = 11 \qquad\qquad x + 2y + 8z = 13$$

$\vec{n}_1 = 3\hat{i} + 4\hat{j} - 9\hat{k}$ $\qquad$ $\vec{n}_2 = \hat{i} + 2\hat{j} + 8\hat{k}$

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\|\,\|\vec{n}_2\|} = \frac{|3 + 8 - 72|}{\sqrt{9+16+81}\,\sqrt{1+4+64}} = \frac{61}{\sqrt{106}\,\sqrt{69}}$$

$$\Rightarrow \boxed{\theta \approx 44.50^\circ}$$

4. (12 points) Consider the plane described by the equation $5x + y - 4z = 9$.

(a) Find a point on the plane. Call it P, and show (or explain) how you know P is on the plane.

$(1, 0, -1)$ IS A POINT ON THE PLANE BECAUSE

$$5(1) + (0) - 4(-1) = 5 + 4 = 9. \checkmark$$

(b) Let $Q\,(-4, -2, 1)$. Show that Q is NOT on the plane.

$$5(-4) + (-2) - 4(1) = -20 - 2 - 4 = -26 \neq 9 \quad \checkmark$$

(c) Let $\vec{n}$ be a vector normal to the plane. Compute $\text{proj}_{\vec{n}}\,\vec{PQ}$.

$\vec{n} = 5\hat{i} + \hat{j} - 4\hat{k}$

$\vec{PQ} = \langle -4-1, -2-0, 1-(-1) \rangle$

$= -5\hat{i} - 2\hat{j} + 2\hat{k}$

$$\text{proj}_{\vec{n}}\,\vec{PQ} = \frac{\vec{PQ}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\,\vec{n} = \frac{-25 - 2 - 8}{25 + 1 + 16}\,\vec{n} = \frac{-35}{42}\,\vec{n} = \boxed{-\frac{5}{6}\vec{n}}$$

(d) Compute $\|\text{proj}_{\vec{n}}\,\vec{PQ}\|$. (You have just computed the distance from Q to the plane.)

$$\|\text{proj}_{\vec{n}}\,\vec{PQ}\| = \frac{5}{6}\|\vec{n}\| = \boxed{\frac{5}{6}\sqrt{42} \approx 5.4}$$

5. (10 points) A triangle has vertices at the points $A\,(1,3,-2)$, $B\,(1,1,-5)$, and $C\,(8,0,-3)$.

(a) Find the area of $\triangle ABC$.

$$\vec{AB} = -2\hat{j} - 3\hat{k}$$
$$\vec{AC} = 7\hat{i} - 3\hat{j} - \hat{k}$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2 & -3 \\ 7 & -3 & -1 \end{vmatrix} = \hat{i}\,(2-9) - \hat{j}\,(0+21) + \hat{k}\,(0+14)$$
$$= -7\hat{i} - 21\hat{j} + 14\hat{k}$$

$$\frac{1}{2}\,\|\vec{AB} \times \vec{AC}\| = \frac{1}{2}\sqrt{49+441+196} = \boxed{\frac{1}{2}\sqrt{686} \approx 13.1}$$

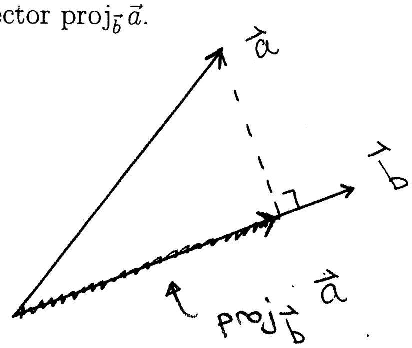
(b) Find an equation of the plane containing A, B, and C.

I'll use $\vec{n} = \hat{i} + 3\hat{j} - 2\hat{k}$

$$x + 3y - 2z = (1) + 3(3) - 2(-2) = 14$$

$$\boxed{x + 3y - 2z = 14}$$

6. (3 points) Sketch two vectors that share a common initial point. Label them $\vec{a}$ and $\vec{b}$. Then sketch and label the vector $\text{proj}_{\vec{b}}\,\vec{a}$.



7. (6 points) Show that the points are collinear. Explain your reasoning.

$$P\,(2,-1,7), \qquad Q\,(17,-10,10), \qquad R\,(-8,5,5)$$

$$\vec{PQ} = 15\hat{i} - 9\hat{j} + 3\hat{k}$$
$$\vec{PR} = -10\hat{i} + 6\hat{j} - 2\hat{k}$$

$\vec{PQ} = -\frac{3}{2}\vec{PR} \Rightarrow$ $\vec{PQ}$ & $\vec{PR}$ ARE PARALLEL.

SINCE $\vec{PQ}$ & $\vec{PR}$ SHARE A COMMON POINT (P) AND ARE PARALLEL, P, Q, R MUST LIE ALONG THE SAME LINE.

8. (6 points) What does it mean for two vectors to be orthogonal? Give an example of two nonzero, orthogonal vectors in 3D-space, and show that your vectors are orthogonal.

VECTORS ARE ORTHOGONAL IFF THEIR DOT PRODUCT IS ZERO.

FOR EXAMPLE, $\vec{u} = 5\hat{i} + 2\hat{j} - 7\hat{k}$ AND $\vec{v} = \hat{i} + \hat{j} + \hat{k}$ ARE ORTHOG:

$\vec{u} \cdot \vec{v} = 5 + 2 - 7 = 0.$

9. (4 points) Suppose you were given two nonzero vectors $\vec{u}$ and $\vec{v}$ in 3D-space. Explain how you could find a nonzero vector that is orthogonal to both $\vec{u}$ and $\vec{v}$.

THE CROSS PRODUCT OF $\vec{u}$ & $\vec{v}$ IS ORTHOG. TO BOTH $\vec{u}$ & $\vec{v}$, UNLESS $\vec{u}$ & $\vec{v}$ ARE PARALLEL.

IF $\vec{u}$ & $\vec{v}$ WERE PARALLEL, THEN $\vec{u} \times \vec{v} = \vec{0}$.
IN THIS CASE, IT SHOULD BE EASY TO FIND AN ORTHOG. VECTOR BY USING DOT PROD. = 0.

10. (6 points) Suppose you were given two nonzero vectors in 3D-space. Briefly describe two different ways that you could test whether the vectors are parallel.

1. THEY ARE PARALLEL IF ONE IS A SCALAR MULTIPLE OF THE OTHER.

2. THEY ARE PARALLEL IF THEIR CROSS PRODUCT IS THE ZERO VECTOR.

3. THEY ARE PARALLEL IF THE ANGLE BETWEEN THEM IS $0°$ OR $180°$, IN WHICH CASE $|\vec{u} \cdot \vec{v}| = \|\vec{u}\| \|\vec{v}\|$.

11. (6 points) Let A, B, and C be arbitrary points in 3-space. Prove that the sum $\vec{AB} + \vec{BC} + \vec{CA}$ does not depend on the points themselves.

$\vec{AB} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$

$\vec{BC} = \langle c_1 - b_1, c_2 - b_2, c_3 - b_3 \rangle$

$\vec{CA} = \langle a_1 - c_1, a_2 - c_2, a_3 - c_3 \rangle.$

ADD THEM TO GET $\langle 0, 0, 0 \rangle$,

WHICH IS INDEPENDENT OF THE POINTS.

ANOTHER APPROACH...

$\vec{AB} + \vec{BC} = \vec{AC}$

AND

$\vec{CA} = -\vec{AC}.$

THEREFORE,

$\underbrace{\vec{AB} + \vec{BC}} + \vec{CA} =$

$\vec{AC} + (-\vec{AC}) = \vec{0}$

12. (6 points) Find a set of parametric equations for the line **segment** connecting $R(3, -4, -2)$ and $S(4, 3, 9)$.

$\vec{RS} = (4-3)\hat{i} + (3-(-4))\hat{j} + (9-(-2))\hat{k}$

$= \hat{i} + 7\hat{j} + 11\hat{k}$

USING $\vec{RS}$ AND R...

$$\boxed{\begin{aligned} x &= t+3 \\ y &= 7t-4 \\ z &= 11t-2 \end{aligned} \qquad 0 \le t \le 1}$$

13. (2 points) Referring to the line segment RS above, find its midpoint.

EITHER

$\left(\frac{3+4}{2}, \frac{-4+3}{2}, \frac{-2+9}{2} \right)$

$= \boxed{\left(\frac{7}{2}, -\frac{1}{2}, \frac{7}{2} \right)}$

OR

$t = \frac{1}{2}$

$\Downarrow$

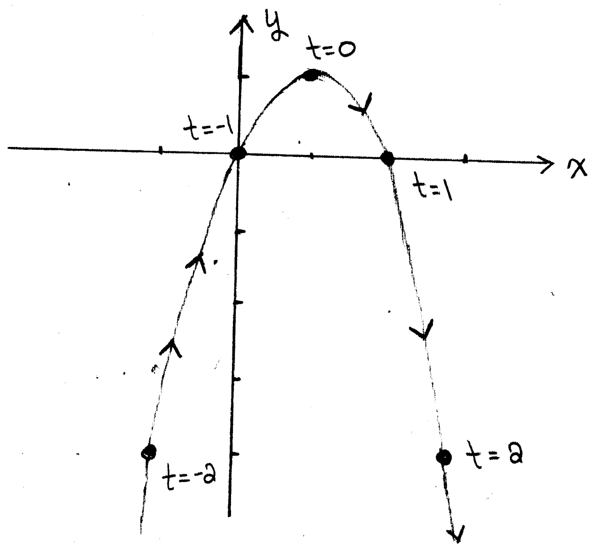
$\left(\frac{7}{2}, -\frac{1}{2}, \frac{7}{2} \right)$

$x = 1 + t \Rightarrow t = x - 1$

$y = 1 - t^2 = 1 - (x-1)^2$

14. (8 points) Plot five points on the graph of $\vec{r}(t) = (1+t)\,\hat{\imath} + (1-t^2)\,\hat{\jmath}$. Then sketch the graph. Draw arrows on your graph to indicate the curve's orientation.

t	(x,y)
-2	$(-1,-3)$
-1	$(0,0)$
0	$(1,1)$
1	$(2,0)$
2	$(3,-3)$



THE GRAPH IS THE GRAPH OF $y = 1 - (x-1)^2$

15. (3 points) Look back at the problem above. Now describe the graph of $\vec{R}(t) = (1+t)\,\hat{\imath} + (1-t^2)\,\hat{\jmath} + t\,\hat{k}$.

SAME GRAPH AS IN #14 BUT TILTED INTO THE $z = x - 1$ PLANE. THAT IS, IT TILTS OUT TO THE RIGHT OF $x = 1$ AND TILTS IN TO THE LEFT OF $x = 1$.

16. (6 points) Let $\vec{r}(t) = \sqrt{t-1}\,\hat{\imath} + \dfrac{\ln t}{t-5}\,\hat{\jmath} + \cos(\pi t)\,\hat{k}$.

(a) Determine the domain of $\vec{r}$.

DOMAIN $= \{t : t \geq 1,\ t > 0, \text{ AND } t \neq 5\}$

$= \boxed{\{t : t \geq 1 \text{ AND } t \neq 5\}}$

(b) Compute $\lim\limits_{t \to 17} \vec{r}(t)$. Write your answer in exact form.

$$\lim_{t \to 17} \left(\sqrt{t-1}\,\hat{\imath} + \frac{\ln t}{t-5}\,\hat{\jmath} + \cos \pi t\,\hat{k} \right)$$

$$= \sqrt{16}\,\hat{\imath} + \frac{\ln 17}{12}\,\hat{\jmath} + \cos 17\pi\,\hat{k}$$

$$= \boxed{4\hat{\imath} + \frac{\ln 17}{12}\,\hat{\jmath} - \hat{k}}$$