Math 233 - Final Exam A

December 6, 2024

Name key

Score ____________

Show all work to receive full credit. Supply explanations where necessary. This portion of the test is due December 12. You must work individually.

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1. (10 points) The temperature at the point (x, y) on a metal plate is given by

$$T(x, y) = 4x^2 - 4xy + y^2.$$

An ant walks on the circle of radius 5 centered at the origin. Use Lagrange multipliers to find the highest and lowest temperatures encountered by the ant. (Hint: In your partial derivative equations, you may find it useful to solve for a variable in one equation and then substitute into the other. After that you should be able to factor.)

$T(x,y) = 4x^2 - 4xy + y^2$

s.t. $x^2 + y^2 = 25$

$$\left.\begin{array}{lll} 8x - 4y = \lambda 2x & \Rightarrow & 4y = 8x - \lambda 2x \\ -4x + 2y = \lambda 2y & \Rightarrow & -8x + 4y = \lambda 4y \\ x^2 + y^2 = 25 & & \end{array}\right\}$$

$$-8x + (8x - \lambda 2x) = \lambda(8x - \lambda 2x)$$

$$-\lambda 2x - \lambda 8x + \lambda^2 2x = 0$$

$$-10\lambda x + 2\lambda^2 x = 0$$

$$2\lambda x(-5 + \lambda) = 0$$

$\lambda = 0$, $x = 0$, $\lambda = 5$

$\lambda = 0$:

$8x = 4y$

$y = 2x$

$x^2 + 4x^2 = 25$

$x^2 = 5$

$x = \pm\sqrt{5}$

$y = \pm 2\sqrt{5}$

$x = 0$:

$y = \pm 5$

BUT THIS CANNOT SATISFY THE CONDITIONS FOR ANY REAL λ

$\lambda = 5$:

$4y = -2x$

$y = -\frac{1}{2}x$

$x^2 + \frac{1}{4}x^2 = 25$

$\frac{5}{4}x^2 = 25$

$x^2 = 20$

$x = \pm 2\sqrt{5}$

$y = \mp\sqrt{5}$

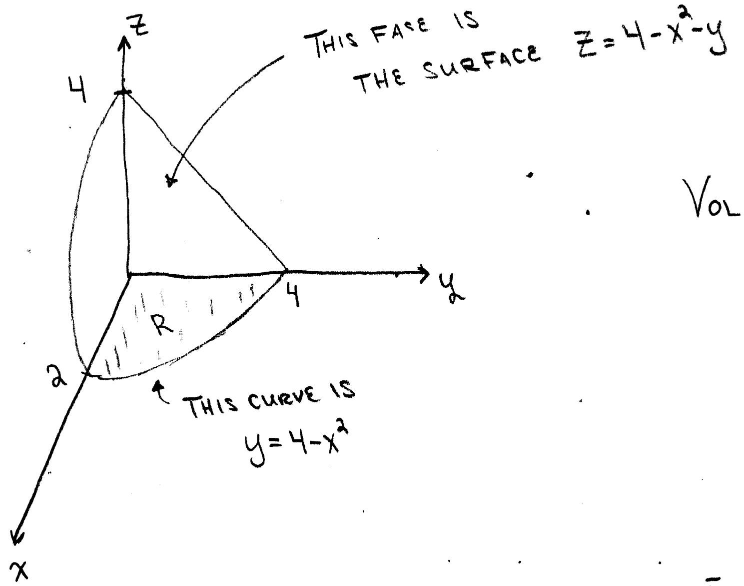
$(\pm\sqrt{5}, \pm 2\sqrt{5})$

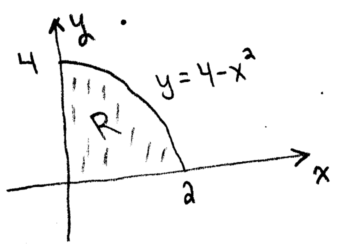
$(\pm 2\sqrt{5}, \mp\sqrt{5})$

$$\boxed{\left.\begin{array}{l} T(\sqrt{5}, 2\sqrt{5}) = 0 \\ T(-\sqrt{5}, -2\sqrt{5}) = 0 \end{array}\right\} \text{MIN TEMP}}$$

$$\boxed{\left.\begin{array}{l} T(2\sqrt{5}, -\sqrt{5}) = 125 \\ T(-2\sqrt{5}, \sqrt{5}) = 125 \end{array}\right\} \text{MAX TEMP}}$$

2. (10 points) The space region W lies in the 1st octant bounded by the coordinate planes ($x = 0$, $y = 0$, and $z = 0$) and the surface $z = 4 - x^2 - y$. Set up and evaluate the triple integrals required to find the average value of $f(x, y, z) = 1 + x^2$ over W. You may use technology to evaluate one of the required integrals, but you must evaluate the other by hand.





$$\mathrm{Vol}(W) = \iiint_W dV$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=4-x^2}\int_{z=0}^{4-x^2-y} 1\,dz\,dy\,dx$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=4-x^2} (4-x^2-y)\,dy\,dx$$

$$= \int_0^2 \left(4y - x^2y - \tfrac{1}{2}y^2\right)\Big|_{y=0}^{y=4-x^2} dx$$

$$= \int_0^2 4(4-x^2) - x^2(4-x^2) - \tfrac{1}{2}(4-x^2)^2\,dx$$

$$= \int_0^2 \left(16 - 4x^2 - 4x^2 + x^4 - 8 + 4x^2 - \tfrac{1}{2}x^4\right) dx$$

$$= \int_0^2 \left(8 - 4x^2 + \tfrac{1}{2}x^4\right) dx = 8x - \tfrac{4}{3}x^3 + \tfrac{1}{10}x^5\Big|_0^2$$

$$= 16 - \frac{32}{3} + \frac{32}{10} = \frac{128}{15}$$

$$\int_{x=0}^{x=2}\int_{y=0}^{y=4-x^2}\int_{z=0}^{z=4-x^2-y} (1+x^2)\,dz\,dy\,dx$$

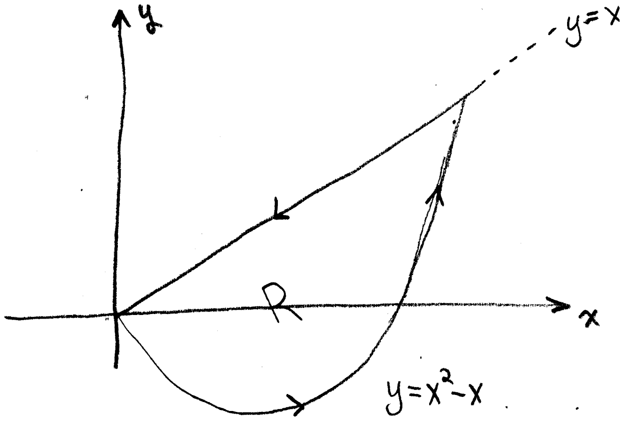
$$= \frac{1408}{105}$$

$$\text{Avg VALUE} = \frac{1408/105}{128/15} = \frac{11}{7}$$

3. (10 points) Let C be the positively-oriented boundary of the plane region enclosed by the graphs of $y = x^2 - x$ and $y = x$. Use Green's theorem to evaluate

$$\int_C -2x^3y^2\,dx + x^4y\,dy.$$

Show all work.



$$x^2 - x = x$$
$$x^2 - 2x = 0$$
$$x(x-2) = 0$$
$$x = 0,\ x = 2$$

$$\int_C -2x^3y^2\,dx + x^4y\,dy$$

$$= \iint_R \left(4x^3y + 4x^3y\right) dA$$

$$= \int_{x=0}^{x=2} \int_{y=x^2-x}^{y=x} 8x^3y\,dy\,dx$$

$$= \int_0^2 4x^3y^2 \Big|_{x^2-x}^{x} dx$$

$$= \int_0^2 4x^5 - 4x^3(x^2-x)^2\,dx = \int_0^2 \left(\cancel{4x^5} - 4x^7 + 8x^6 - \cancel{4x^5}\right) dx$$

$$= -\frac{1}{2}x^8 + \frac{8}{7}x^7 \Big|_0^2 = -\frac{1}{2}(2)^8 + \frac{8}{7}(2)^7$$

$$= \boxed{\frac{128}{7}}$$