Math 173 - Test 3a

April 23, 2015

Name key

Score ________

Show all work to receive full credit. Supply explanations where necessary.

1. (8 points) Find and classify the critical points of $f(x, y) = 12x^2 + y^3 - 12xy$. Determine all relative extreme values and locate any of the graph's saddle points.

$f_x(x,y) = 24x - 12y = 0 \Rightarrow 24x = 12y \Rightarrow y = 2x$

$f_y(x,y) = 3y^2 - 12x = 0$

$12x^2 - 12x = 0$

$12x(x-1) = 0$

$x = 0 \quad x = 1$

$\Downarrow \quad \Downarrow$

$y = 0 \quad y = 2$

$(0,0) \quad (1,2)$

$f_{xx}(x,y) = 24$

$f_{xy}(x,y) = -12$

$f_{yy}(x,y) = 6y$

$$D(x,y) = \begin{vmatrix} 24 & -12 \\ -12 & 6y \end{vmatrix} = 144y - 144$$

$(0,0)$: $D(0,0) = -144$, $f(0,0) = 0$

$\Rightarrow \boxed{(0,0,0) \text{ IS A SADDLE PT}}$

$(1,2)$: $D(1,2) = 144(2) - 144$
$= 144$, $f_{xx}(1,2) = 24 > 0$, $f(1,2) = -4$

$\Rightarrow \boxed{f(1,2) = -4 \text{ IS A RELATIVE MIN}}$

2. (8 points) Find an equation of the plane tangent to the graph of $z = ye^{2xy}$ at the point $(0, 2, 2)$.

$F(x,y,z) = z - ye^{2xy}$ NOW OUR SURFACE IS THE LEVEL SURFACE $F(x,y,z) = 0$

$$\vec{\nabla} F(x,y,z) = -2y^2 e^{2xy}\,\hat{i} + \left(-e^{2xy} - 2xye^{2xy}\right)\hat{j} + \hat{k}$$

$$\vec{N} = \vec{\nabla} F(0,2,2) = -8\hat{i} - \hat{j} + \hat{k} \quad (\text{I'LL USE } 8\hat{i} + \hat{j} - \hat{k}.)$$

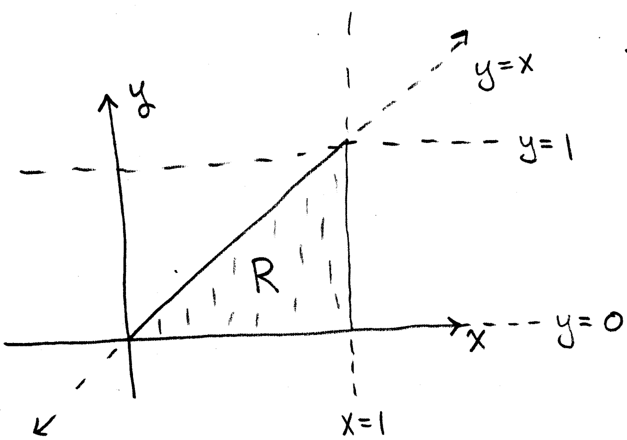
TANGENT PLANE IS

$$8(x-0) + (y-2) - (z-2) = 0$$

OR $\boxed{8x + y - z = 0}$

3. (8 points) Sketch the region of integration, reverse the order, and evaluate. Do not use your calculator to evaluate any integrals.

$$\int_0^1 \int_y^1 \sin x^2 \, dx \, dy$$



$$\int_{x=0}^{x=1} \int_{y=0}^{y=x} \sin x^2 \, dy \, dx$$

$$= \int_0^1 y \sin x^2 \Big|_{y=0}^{y=x} dx$$

$$= \int_0^1 x \sin x^2 \, dx$$

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2} du = x\,dx$

$$\frac{1}{2} \int_{u=0}^{u=1} \sin u \, du$$

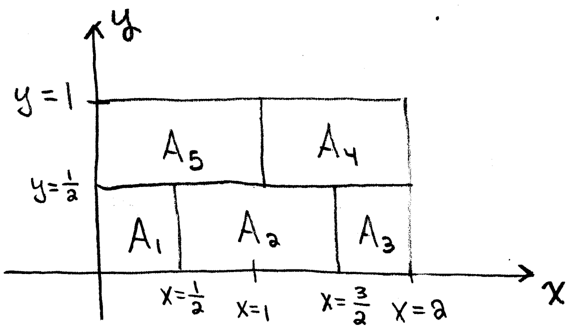
$$= \frac{1}{2}(-\cos u)\Big|_{u=0}^{u=1} = \frac{1}{2}(-\cos 1 + 1)$$

$$= \boxed{\frac{1}{2}(1 - \cos 1) \approx 0.2298}$$

4. (8 points) Consider the double integral

$$\iint_R (x+y)\,dA,$$

where R is the rectangular region between the graphs of $x=0$, $x=2$, $y=0$, and $y=1$. Use a Riemann sum with 5 subregions to estimate the value of the double integral.



$f(x,y) = x+y$

A_1: Area $= \frac{1}{4}$, point $\left(\frac{1}{4}, \frac{1}{4}\right)$

A_2: Area $= \frac{1}{2}$, point $\left(1, \frac{1}{4}\right)$

A_3: Area $= \frac{1}{4}$, point $\left(\frac{7}{4}, \frac{1}{4}\right)$

A_4: Area $= \frac{1}{2}$, point $\left(\frac{3}{2}, \frac{3}{4}\right)$

A_5: Area $= \frac{1}{2}$, point $\left(\frac{1}{2}, \frac{3}{4}\right)$

Riemann sum

$$= f\left(\tfrac{1}{4},\tfrac{1}{4}\right)\left(\tfrac{1}{4}\right) + f\left(1,\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right) + f\left(\tfrac{7}{4},\tfrac{1}{4}\right)\left(\tfrac{1}{4}\right) + f\left(\tfrac{3}{2},\tfrac{3}{4}\right)\left(\tfrac{1}{2}\right) + f\left(\tfrac{1}{2},\tfrac{3}{4}\right)\left(\tfrac{1}{2}\right)$$

$$= \left(\tfrac{2}{4}\right)\left(\tfrac{1}{4}\right) + \left(\tfrac{5}{4}\right)\left(\tfrac{1}{2}\right) + \left(\tfrac{8}{4}\right)\left(\tfrac{1}{4}\right) + \left(\tfrac{9}{4}\right)\left(\tfrac{1}{2}\right) + \left(\tfrac{5}{4}\right)\left(\tfrac{1}{2}\right)$$

$$= \frac{2+10+8+18+10}{16} = \frac{48}{16} = \boxed{3}$$

5. (8 points) Use Lagrange multipliers to find the extreme values of

$$f(x,y) = x^2 + 2y^2 - 2x + 3$$

subject to the constraint $\underbrace{x^2 + y^2}_{g(x,y)} = 9$.

$2x - 2 = \lambda 2x$

$4y = \lambda 2y \Rightarrow 4y - \lambda 2y = 0 \Rightarrow 2y(2-\lambda) = 0$

$x^2 + y^2 = 9$

$y = 0.$ or $\lambda = 2$

$x = \pm 3$

$2x - 2 = 4x$

$-2 = 2x$

$x = -1.$

$y = \pm\sqrt{8}$

$(3,0)$

$(-3,0)$

$(-1, \sqrt{8})$

$(-1, -\sqrt{8})$

$f(3,0) = 6 \leftarrow$ Minimum value

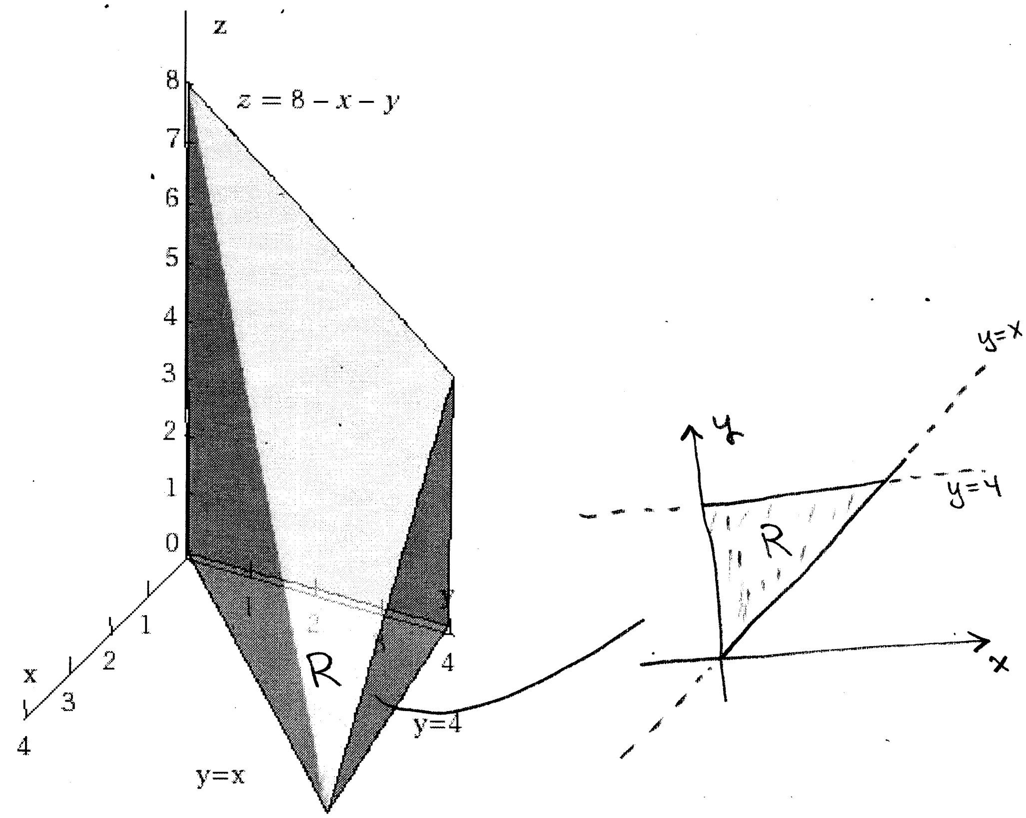
$f(-3,0) = 18$

$f(-1,\sqrt{8}) = 22$

$f(-1,-\sqrt{8}) = 22$

$\leftarrow$ Maximum value

6. (10 points) Use a double integral to find the volume of the region in the 1st octant bounded by the planes $y = x$, $y = 4$, $x = 0$, and $x + y + z = 8$.



$$\text{Volume} = \int_{y=0}^{y=4} \int_{x=0}^{x=y} (8-x-y)\,dx\,dy = \int_0^4 \left(8y - \tfrac{1}{2}y^2 - y^2\right) dy$$

$$= 4y^2 - \tfrac{1}{6}y^3 - \tfrac{1}{3}y^3 \Big|_0^4 = 64 - \frac{64}{6} - \frac{64}{3}$$

$$= \boxed{32}$$

Math 173 - Test 3b

April 23, 2015

Name key

Score ______

Show all work to receive full credit. Supply explanations where necessary. This portion of the test is due Monday, April 27. YOU MUST WORK INDIVIDUALLY ON THIS TEST—YOU WILL NOT BE GIVEN ANY CREDIT FOR GROUP WORK.

1. (8 points) Find the linearization of $h(x, y) = \ln(x^2 + 2xy + 2y^2)$ at the point where $(x, y) = (1, 1)$. Then use your linearization to approximate $h(0.98, 1.01)$.

$$L(x,y) = h(1,1) + h_x(1,1)(x-1) + h_y(1,1)(y-1)$$

$$h_x(x,y) = \frac{2x+2y}{x^2+2xy+2y^2} \qquad h_y(x,y) = \frac{2x+4y}{x^2+2xy+2y^2}$$

$$h_x(1,1) = 4/5$$

$$h_y(1,1) = 6/5$$

$$h(1,1) = \ln 5$$

$\Rightarrow$ LINEARIZATION IS

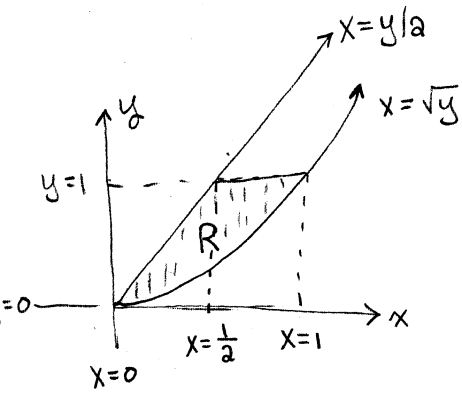
$$L(x,y) = \ln 5 + \frac{4}{5}(x-1) + \frac{6}{5}(y-1)$$

$$h(0.98, 1.01) \approx L(0.98, 1.01) = \ln 5 - \frac{0.08}{5} + \frac{0.06}{5} \approx 1.6054$$

2. (8 points) Sketch the region of integration, reverse the order, and evaluate. Do not use your calculator to evaluate any integrals.

$$\int_0^1 \int_{y/2}^{\sqrt{y}} (4x + 2)\, dx\, dy$$

ORIGINAL ORDER IS EASIER TO EVALUATE!



$$\int_{x=0}^{x=1/2} \int_{y=x^2}^{y=2x} (4x+2)\, dy\, dx + \int_{x=1/2}^{x=1} \int_{y=x^2}^{y=1} (4x+2)\, dy\, dx =$$

$$\int_0^{1/2} 4xy + 2y \Big|_{y=x^2}^{y=2x} dx + \int_{1/2}^1 4xy + 2y \Big|_{y=x^2}^{y=1} dx = \int_0^{1/2} (8x^2 + 4x - 4x^3 - 2x^2)\, dx + \int_{1/2}^1 (4x + 2 - 4x^3 - 2x^2)\, dx$$

$$= \int_0^{1/2} 8x^2\, dx + \int_{1/2}^1 2\, dx + \int_0^1 (4x - 4x^3 - 2x^2)\, dx = \frac{1}{3} + 1 + \frac{4}{2} - \frac{4}{4} - \frac{2}{3} = \boxed{\frac{5}{3}}$$

3. (8 points) Find and classify the critical points of $f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy$. Determine all relative extreme values and locate any of the graph's saddle points.

$f_x(x,y) = -6x + 6y = 0 \Rightarrow y = x$

$f_y(x,y) = 6y - 6y^2 + 6x = 0 \quad 12y - 6y^2 = 0 \Rightarrow 6y(2-y) = 0$

$y = 0, \ y = 2$

CRITICAL PTS ARE $(0,0)$ AND $(2,2)$.

$f_{xx}(x,y) = -6$

$f_{xy}(x,y) = 6$

$f_{yy}(x,y) = 6 - 12y$

$$d(x,y) = \begin{vmatrix} -6 & 6 \\ 6 & 6-12y \end{vmatrix} = 72y - 72$$

$d(0,0) = -72$ & $f(0,0) = 0$

$\Rightarrow$ $\boxed{(0,0,0) \text{ IS A SADDLE PT.}}$

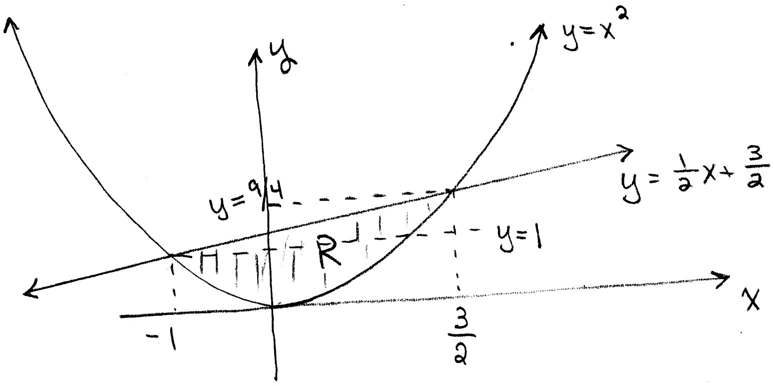
$d(2,2) = 72$ & $f_{xx}(2,2) = -6 < 0$

$\Rightarrow$ $\boxed{f(2,2) = 8 \text{ IS A RELATIVE MAX}}$

4. (12 points) Consider the double integral

$$\iint_R y\, dA,$$

where R is the region in the xy-plane between the graphs of $y = x^2$ and $y = \frac{1}{2}x + \frac{3}{2}$. Sketch the region of integration, reverse the order of integration, and evaluate the double integral (using either order of integration).



$x = 2y-3$

$$\frac{1}{2}x + \frac{3}{2} = x^2 \Rightarrow 2x^2 - x - 3 = 0$$

$$(2x-3)(x+1) = 0$$

$$x = \frac{3}{2} \quad x = -1$$

$$\int_{x=-1}^{x=3/2} \int_{y=x^2}^{y=\frac{1}{2}x+\frac{3}{2}} y\, dy\, dx = \int_{y=0}^{y=1} \int_{x=-\sqrt{y}}^{x=\sqrt{y}} y\, dx\, dy + \int_{y=1}^{y=9/4} \int_{x=2y-3}^{x=\sqrt{y}} y\, dx\, dy$$

$$\int_{-1}^{3/2} \frac{1}{2}y^2 \Big|_{y=x^2}^{y=\frac{1}{2}x+\frac{3}{2}} dx = \frac{1}{2}\int_{-1}^{3/2} \left(\frac{1}{2}x+\frac{3}{2}\right)^2 - x^4\, dx$$

$$= \frac{1}{2}\int_{-1}^{3/2} \left(\frac{1}{4}x^2 + \frac{3}{2}x + \frac{9}{4} - x^4\right) dx = \frac{1}{2}\left(\frac{1}{12}x^3 + \frac{3}{4}x^2 + \frac{9}{4}x - \frac{1}{5}x^5\right)\Bigg|_{-1}^{3/2}$$

$$= \frac{1}{2}\left[\frac{27}{96} + \frac{27}{16} + \frac{27}{8} - \frac{243}{160} + \frac{1}{12} - \frac{3}{4} + \frac{9}{4} - \frac{1}{5}\right]$$

$$= \boxed{\frac{125}{48}}$$

5. (10 points) The temperature at the point (x, y) on a metal plate is given by

$$T(x,y) = 4x^2 - 4xy + y^2.$$

$x^2 + y^2 = 25$

An ant on the plate walks around the circle of radius 5 centered at the origin. Find the highest and lowest temperatures encountered by the ant.

Lagrange Multipliers...

$$\left.\begin{aligned} 8x - 4y &= \lambda 2x \\ -4x + 2y &= \lambda 2y \end{aligned}\right\}$$

$$x^2 + y^2 = 25$$

$$\begin{aligned} 8x - 4y &= \lambda 2x \\ -8x + 4y &= \lambda 4y \\ \hline 0 &= \lambda(2x + 4y) \end{aligned}$$

$\lambda = 0$ or $x = -2y$

$\lambda = 0 \rightarrow y = 2x$, $x^2 + 4x^2 = 25$, $x = \pm\sqrt{5}$ $\rightarrow$ $(\sqrt{5}, 2\sqrt{5})$, $(-\sqrt{5}, -2\sqrt{5})$

$x = -2y \rightarrow 4y^2 + y^2 = 25$, $y = \pm\sqrt{5}$ $\rightarrow$ $(-2\sqrt{5}, \sqrt{5})$, $(2\sqrt{5}, -\sqrt{5})$

$$\boxed{\begin{aligned} &\left.\begin{aligned} f(\sqrt{5}, 2\sqrt{5}) &= 0 \\ f(-\sqrt{5}, -2\sqrt{5}) &= 0 \end{aligned}\right\} \text{MIN VALUE} \\ &\left.\begin{aligned} f(-2\sqrt{5}, \sqrt{5}) &= 125 \\ f(2\sqrt{5}, -\sqrt{5}) &= 125 \end{aligned}\right\} \text{MAX VALUE} \end{aligned}}$$

6. (4 points) Refer back to problem 4. Find the average value of $f(x, y) = y$ over R. (See page 982 of the text.)

$$\text{Area of } R = \int_{-1}^{3/2} \left(\tfrac{1}{2}x + \tfrac{3}{2} - x^2\right) dx = \tfrac{1}{4}x^2 + \tfrac{3}{2}x - \tfrac{1}{3}x^3 \Big|_{-1}^{3/2}$$

$$= \frac{9}{16} + \frac{9}{4} - \frac{27}{24} - \frac{1}{4} + \frac{3}{2} - \frac{1}{3} = \frac{125}{48}$$

$$\text{Average value} = \frac{1}{\text{Area of } R} \iint_R y \, dA = \frac{1}{125/48}\left(\frac{125}{48}\right) = \boxed{1}$$