Math 173 - Final Exam

May 12, 2014

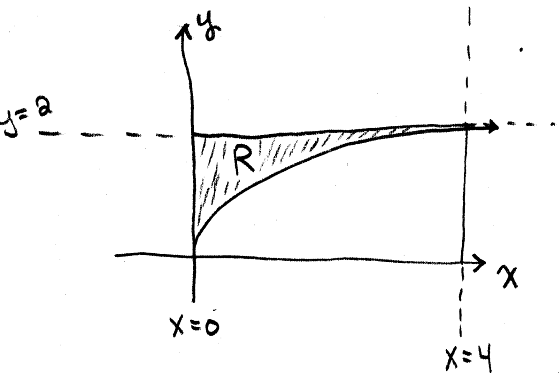
Name key

Score

Show all work. Supply explanations when necessary. Unless otherwise indicated, each problem is worth 12 points.

1. Sketch the region of integration, reverse the order of integration, and **evaluate the iterated integral by hand.**

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{x}{y^5+1}\,dy\,dx$$



$$\int_{y=0}^{y=2}\int_{x=0}^{x=y^2} \frac{x}{y^5+1}\,dx\,dy = \frac{1}{2}\int_{y=0}^{y=2} \frac{y^4}{y^5+1}\,dy$$

$u = y^5+1 \quad du = 5y^4\,dy$

$$= \frac{1}{10}\int_{u=1}^{u=33} \frac{1}{u}\,du = \boxed{\frac{1}{10}\ln 33}$$

2. Find and classify all relative extreme values and saddle points.

$$f(x,y) = x^2 + xy + 2y^2 + x - 3y + 10$$

$f_x(x,y) = 2x + y + 1 = 0$

$f_y(x,y) = x + 4y - 3 = 0$

$-1\,(2x + y = -1)$

$+2\,(x + 4y = 3)$

$7y = 7$

$y = 1$

$x = -1$

ONLY CRITICAL PT IS $(-1,1)$

$$D(x,y) = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 7$$

$D(-1,1) = 7 > 0$ AND $f_{xx}(-1,1) = 2 > 0$

$\Rightarrow f(-1,1) = 8$ IS A RELATIVE MIN

3. A plane passes through the points $P\,(2,1,1)$, $Q\,(0,4,1)$ and $R\,(-2,1,4)$. Find a set of parametric equations for the line normal to the plane and passing through $(5,-9,3)$.

$\vec{PQ} = -2\hat{i} + 3\hat{j}$

$\vec{PR} = -4\hat{i} + 3\hat{k}$

$$\vec{N} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 3 & 0 \\ -4 & 0 & 3 \end{vmatrix}$$

$= 9\hat{i} + 6\hat{j} + 12\hat{k}$

DIRECTION: $9\hat{i} + 6\hat{j} + 12\hat{k}$

POINT: $(5,-9,3)$

$$\boxed{\begin{aligned} x &= 9t+5 \\ y &= 6t-9 \\ z &= 12t+3 \end{aligned}}$$

4. The curves described by the vector-valued functions $\vec{r_1}(t) = t\hat{\imath} + (t^2+3)\hat{\jmath} + (6-5t)\hat{k}$ and $\vec{r_2}(t) = t^2\hat{\imath} + (2t+2)\hat{\jmath} + t^3\hat{k}$ intersect at the point $(1,4,1)$. Find the angle of intersection of the curves. (Hint: Find the angle between the velocity vectors at the point of intersection.)

THE POINT $(1,4,1)$ CORRESPONDS TO $t=1$

$\vec{r_1}'(t) = \hat{i} + 2t\hat{j} - 5\hat{k}$ $\qquad$ $\vec{r_1}'(1) = \hat{i} + 2\hat{j} - 5\hat{k}$

$\vec{r_2}'(t) = 2t\hat{i} + 2\hat{j} + 3t^2\hat{k}$ $\qquad$ $\vec{r_2}'(1) = 2\hat{i} + 2\hat{j} + 3\hat{k}$

$$\cos\theta = \frac{\vec{r_1}'(1) \cdot \vec{r_2}'(1)}{\|\vec{r_1}'(1)\|\,\|\vec{r_2}'(1)\|} = \frac{2+4-15}{\sqrt{30}\,\sqrt{17}} = \frac{-9}{\sqrt{510}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{-9}{\sqrt{510}}\right) \approx 1.98 \approx 113.5^\circ}$$

5. (14 points) Evaluate the limit or show that it does not exist.

(a) $\lim_{(x,y)\to(1,1)} \frac{5x-5y}{y^2-x^2} = \lim_{(x,y)\to(1,1)} \frac{-5(y-x)}{(y-x)(y+x)} = \lim_{(x,y)\to(1,1)} \left(\frac{-5}{y+x}\right)$

$= \boxed{\frac{-5}{2}}$

(b) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$

ALONG $y=0$: $\lim_{x\to 0} \frac{0}{x^4} = \lim_{x\to 0} 0 = 0$

ALONG $y=x^2$: $\lim_{x\to 0} \frac{x^4}{x^4+x^4} = \lim_{x\to 0} \frac{x^4}{2x^4} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$.

LIMIT DNE

6. Suppose w depends on x, y, and z according to $w = x\cos(yz)$ and x, y, and z depend on s and t:

$$x = s^2, \qquad y = t^2, \qquad z = s - 2t.$$

Write the chain rule formula for $\partial w/\partial t$. Then use your formula to find $\partial w/\partial t$ at $(s,t) = (\pi, 0)$.

AT $(s,t) = (\pi, 0)$,

$x = \pi^2$

$y = 0$

$z = \pi$

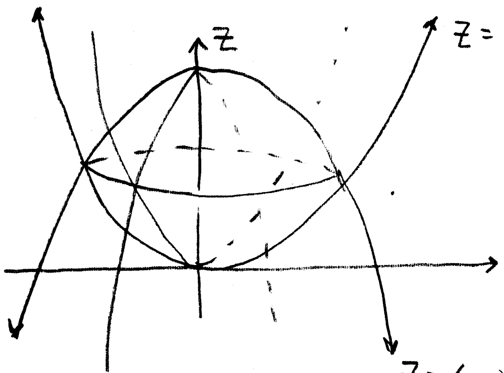
$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$= \cos(yz)(0) + (-xz\sin(yz))(2t) + (-xy\sin(yz))(-2)$$

$$= 0 + 0 + 0 = \boxed{0}$$

7. Let S be the space region bounded by the paraboloids $z = 2x^2+2y^2$ and $z = 6-x^2-y^2$. Evaluate the following triple integral **by hand**. (Hint: It's best to use cylindrical coordinates.)

$$\iiint_S (x^2+y^2)\,dV$$

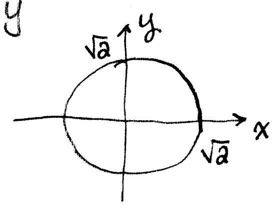


$z = 2x^2+2y^2 = 2r^2$

$z = 6-x^2-y^2 = 6-r^2$

$2x^2+2y^2 = 6-x^2-y^2$

$x^2+y^2 = 2$



$$\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=\sqrt{2}}\int_{z=2r^2}^{z=6-r^2} r^2\, r\, dz\, dr\, d\theta$$

$$= 2\pi \int_0^{\sqrt{2}}\int_{2r^2}^{6-r^2} r^3\, dz\, dr$$

$$= 2\pi \int_0^{\sqrt{2}} (6r^3 - 3r^5)\, dr$$

$$= 2\pi\left(\tfrac{6}{4}r^4 - \tfrac{1}{2}r^6\right)\Big|_0^{\sqrt{2}} = 2\pi(6-4)$$

$$= \boxed{4\pi}$$

8. The quarterback of a football team throws the ball with an initial speed of 54 feet per second, at an angle of 36°, and at a height of 6 feet above the playing field. How far downfield has the ball traveled at the moment when it reaches its maximum height? (Ignore air resistance and use $g = 32\,\text{ft/sec}^2$.)

$$\vec{r}(t) = 54\cos 36^\circ t\,\hat{i} + (-16t^2 + 54\sin 36^\circ t + 6)\,\hat{j}$$

$$\vec{r}\,'(t) = 54\cos 36^\circ\,\hat{i} + (-32t + 54\sin 36^\circ)\,\hat{j}$$

MAX HEIGHT WHEN $t = \dfrac{54\sin 36^\circ}{32} \approx 0.9919$ sec

$$(54\cos 36^\circ)(0.9919) \approx \boxed{43.3 \text{ FT}}$$

9. The temperature at the point (x, y, z) is given by

$$T(x, y, z) = xz^2 \cos(\pi y)$$

where T is measured in °C and x, y, and z in centimeters.

(a) At the point $(2, -1, 2)$, in what direction does the temperature decrease the fastest?

$$\vec{\nabla} T(x,y,z) = z^2 \cos(\pi y)\,\hat{i} - \pi x z^2 \sin(\pi y)\,\hat{j} + 2xz\cos(\pi y)\,\hat{k}$$

$$\vec{\nabla} T(2,-1,2) = -4\hat{i} + 0\hat{j} - 8\hat{k}$$

MAX DECREASE IN DIRECTION OPPOSITE GRADIENT $\Rightarrow \boxed{4\hat{i} + 8\hat{k}}$

(b) Find the rate of change of temperature at the point $(2, -1, 2)$ in the direction toward the point $(3, -3, 3)$.

$$\vec{u} = \hat{i} - 2\hat{j} + \hat{k}$$

$$\|\vec{u}\| = \sqrt{1+4+1} = \sqrt{6}$$

$$\frac{\vec{\nabla} T(2,-1,2) \cdot \vec{u}}{\|\vec{u}\|} = \frac{-4-8}{\sqrt{6}} = \frac{-12}{\sqrt{6}} = \boxed{-2\sqrt{6}}$$

10. Let $\vec{r}(t) = t^2\hat{\imath} + (\sin t - t\cos t)\hat{\jmath} + (\cos t + t\sin t)\hat{k}$. Find the unit tangent vector and the principal unit normal vector. (Hint: If you're doing everything correctly, the computations should not be messy.)

$$\vec{r}\,'(t) = 2t\hat{i} + (t\sin t)\hat{j} + (t\cos t)\hat{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{4t^2 + t^2(\sin^2 t + \cos^2 t)} = \sqrt{5}\,t, \quad t \geq 0$$

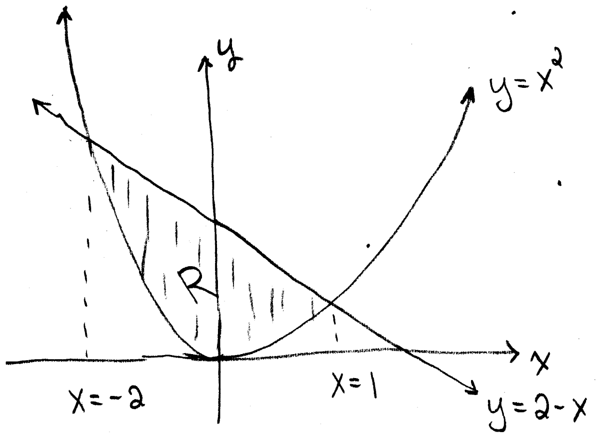
$$\boxed{\hat{T}(t) = \frac{2\hat{i} + \sin t\,\hat{j} + \cos t\,\hat{k}}{\sqrt{5}}}$$

$$\hat{T}'(t) = \frac{\cos t\,\hat{j} - \sin t\,\hat{k}}{\sqrt{5}}$$

$$\boxed{\hat{N}(t) = \cos t\,\hat{j} - \sin t\,\hat{k}}$$

$$\|\hat{T}'(t)\| = \frac{1}{\sqrt{5}}$$

11. (13 points) A thin plate is bounded by the graphs of $y = x^2$ and $y = 2 - x$. The density of the plate at the point (x, y) is given by $\rho(x, y) = 5 + xy$. Sketch the thin plate. Then set up the three iterated integrals required to find the center of mass of the plate. Do not evaluate the integrals.



$$x^2 = 2 - x$$
$$\Rightarrow x^2 + x - 2 = 0$$
$$(x+2)(x-1) = 0$$
$$x = -2, \quad x = 1$$

$$\text{Mass} = M = \int_{x=-2}^{x=1} \int_{y=x^2}^{y=2-x} (5 + xy)\, dy\, dx$$

$$M_y = \int_{x=-2}^{x=1} \int_{y=x^2}^{y=2-x} x(5 + xy)\, dy\, dx$$

$$M_x = \int_{x=-2}^{x=1} \int_{y=x^2}^{y=2-x} y(5 + xy)\, dy\, dx$$

Center of Mass is $\left(\frac{M_y}{M}, \frac{M_x}{M} \right)$.

12. (15 points) Let P and Q be the points $(3, 1, 5)$ and $(-2, 2, 7)$, respectively.

(a) Find a vector of length 4 in the direction of $\overrightarrow{PQ}$.

$$\overrightarrow{PQ} = -5\hat{\imath} + \hat{\jmath} + 2\hat{k}$$

$$\|\overrightarrow{PQ}\| = \sqrt{25+1+4} = \sqrt{30}$$

$$\boxed{\frac{4}{\sqrt{30}}\left(-5\hat{\imath} + \hat{\jmath} + 2\hat{k}\right)}$$

(b) Find a unit vector in the xy-plane that is orthogonal to $\overrightarrow{PQ}$.

$$\vec{u} = \hat{\imath} + 5\hat{\jmath}$$

$$\vec{u} \cdot \overrightarrow{PQ} = -5 + 5 + 0 = 0$$

$$\boxed{\frac{\vec{u}}{\|\vec{u}\|} = \frac{1}{\sqrt{26}}\left(\hat{\imath} + 5\hat{\jmath}\right)}$$

(c) Find the projection of $\overrightarrow{PQ}$ onto $\overbrace{\hat{\imath} + \hat{\jmath} - \hat{k}}^{\vec{v}}$.

$$\operatorname{proj}_{\vec{v}} \overrightarrow{PQ} = \frac{\overrightarrow{PQ} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\, \vec{v}$$

$$= \frac{-5 + 1 - 2}{3}\, \vec{v} = -2\vec{v}$$

$$= \boxed{-2\hat{\imath} - 2\hat{\jmath} + 2\hat{k}}$$