**Math 173 - Test 3a**
April 24, 2014

Name key

Score ____________

Show all work to receive full credit. Supply explanations where necessary.

1. (6 points) Use the appropriate form of the Chain Rule to find $\partial w/\partial t$.

$$w = x\cos yz; \quad x = s^2t,\ y = t^3,\ z = s - 2t$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$= (\cos yz)(s^2) + (-xz\sin yz)(3t^2) + (-xy\sin yz)(-2)$$

$$= \boxed{s^2\cos yz + (2y - 3zt^2)(x\sin yz)}$$

2. (6 points) Let $f(x, y, z) = xy^2z^4$. Find the maximum value of the directional derivative of $f$ at the point $(2, 1, 1)$.

MAX DIRECTIONAL DERIVATIVE AT $(2,1,1)$

$$= \|\vec{\nabla} f(2,1,1)\|$$

$$\vec{\nabla} f(x,y,z) = y^2z^4\,\hat{i} + 2xyz^4\,\hat{j} + 4xy^2z^3\,\hat{k}$$

$$\vec{\nabla} f(2,1,1) = \hat{i} + 4\hat{j} + 8\hat{k}$$

$$\|\vec{\nabla} f(2,1,1)\| = \sqrt{1+16+64} = \sqrt{81} = \boxed{9}$$

3. (6 points) Suppose $y$ is implicitly defined as a function of $x$ by the equation

$$4y^2 = x^2y^2 - 9x^2.$$

Find $dy/dx$ at $(-4, 2\sqrt{3})$.

$$F(x,y) = 4y^2 - x^2y^2 + 9x^2$$

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-(-2xy^2 + 18x)}{8y - 2x^2y} = \frac{2xy^2 - 18x}{8y - 2x^2y}$$

$$\left.\frac{dy}{dx}\right|_{(-4,2\sqrt{3})} = \frac{-96 + 72}{16\sqrt{3} - 64\sqrt{3}} = \frac{-24}{-48\sqrt{3}} = \boxed{\frac{1}{2\sqrt{3}}}$$

4. (4 points) Suppose $z = f(x, y)$ is a differentiable function and $(x_0, y_0, z_0)$ is a point on its graph. Is it true that $\vec{\nabla} f(x_0, y_0)$ is normal to the graph of $f$ at $(x_0, y_0, z_0)$? Explain your reasoning.

NO, THE GRADIENT IS NORMAL TO LEVEL CURVES/SURFACES.

THESE ARE TRUE:

① $\vec{\nabla} f(x_0, y_0)$ IS NORMAL TO THE LEVEL CURVE $f(x,y) = z_0$ AT THE POINT $(x_0, y_0)$

② LET $F(x,y,z) = z - f(x,y)$. $\vec{\nabla} F(x,y,z)$ IS NORMAL TO THE GRAPH OF $f$ AT $(x_0, y_0, z_0)$.

5. (8 points) Find and classify the critical points of $f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3$.

$f_x(x,y) = 4x + 2y + 2 = 0$

$f_y(x,y) = 2x + 2y = 0 \Rightarrow x = -y \Rightarrow 4x + 2y + 2 = 0$

$\Rightarrow 4x - 2x + 2 = 0$

$\Rightarrow x = -1$

$\Rightarrow y = 1$

$$D(x,y) = \begin{vmatrix} 4 & 2 \\ 2 & 2 \end{vmatrix} = 8 - 4 = 4$$

$D(-1,1) = 4 > 0$

AND $f_{xx}(-1,1) = 4 > 0$

$\Rightarrow$ $f(-1,1) = -4$ IS A RELATIVE MINIMUM

6. (8 points) Find the extreme values of $f(x, y) = 8x + 15y$ subject to $x^2 + y^2 = 289$.

$f(x,y) = 8x + 15y$

$g(x,y) = x^2 + y^2$

$\vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y)$

$g(x,y) = 289$

$\Downarrow$

$8 = 2\lambda x \Rightarrow x = \frac{8}{2\lambda}$

$15 = 2\lambda y \Rightarrow y = \frac{15}{2\lambda}$

$x^2 + y^2 = 289$

$\frac{64}{4\lambda^2} + \frac{225}{4\lambda^2} = 289$

$\Downarrow$

$\lambda^2 = \frac{1}{4}$

$\lambda = \pm \frac{1}{2}$

$\lambda = \frac{1}{2}$ $\Downarrow$ $x = 8$, $y = 15$

$\lambda = -\frac{1}{2}$ $\Downarrow$ $x = -8$, $y = -15$

CRITICAL POINTS ARE

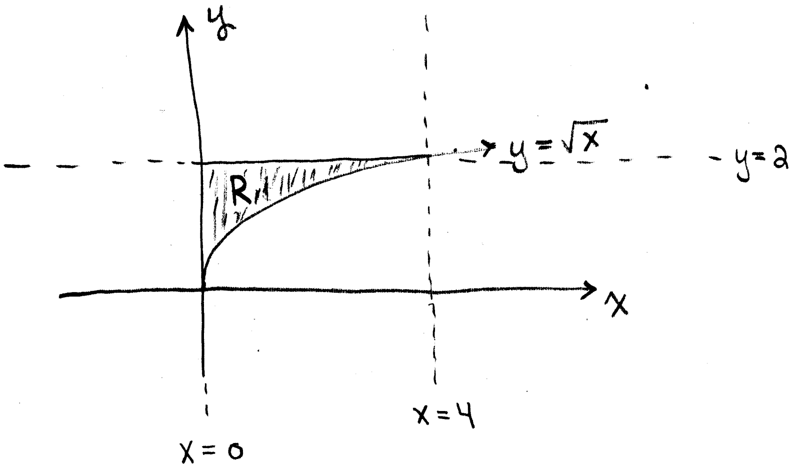
$(8, 15)$ AND $(-8, -15)$

$f(8,15) = 289$ CONSTRAINED MAX

$f(-8,-15) = -289$ CONSTRAINED MIN

7. (12 points) Sketch the region of integration, reverse the order of integration, and evaluate your new iterated integral (by hand).

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{3}{2+y^3}\, dy\, dx$$




$$\int_{y=0}^{y=2} \int_{x=0}^{x=y^2} \frac{3}{2+y^3}\, dx\, dy = \int_{y=0}^{y=2} \frac{3x}{2+y^3}\Bigg|_{x=0}^{x=y^2} dy$$

$$= \int_{y=0}^{y=2} \frac{3y^2}{2+y^3}\, dy$$

$u = 2+y^3$

$du = 3y^2\, dy$

$$\int_{u=2}^{u=10} \frac{1}{u}\, du = \ln 10 - \ln 2$$

$$= \boxed{\ln 5}$$

# Math 173 - Test 3b

April 24, 2014

Name key

Score ______

Show all work to receive full credit. Supply explanations where necessary. This portion of the test is due Monday, April 28. You must work individually on this test.

1. (5 points) Suppose $z$ is implicitly defined as a function of $x$ and $y$ by the equation

$$x \ln y + y^2 z + z^3 = 8.$$

Find $\partial z/\partial y$.

LET $F(x,y,z) = x \ln y + y^2 z + z^3$.

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-\left[\frac{x}{y} + 2yz\right]}{y^2 + 3z^2} = \boxed{\frac{-x - 2y^2 z}{y^3 + 3yz^2}}$$

2. (5 points) Find an equation of the plane tangent to the surface $x = y(2z - 3)$ at the point $(4, 4, 2)$.

$F(x,y,z) = x - 2yz + 3y$

THE SURFACE IS THE LEVEL SURFACE $F(x,y,z) = 0$, PASSING THROUGH $(4,4,2)$

$\vec{\nabla} F(x,y,z) = \hat{i} + (3 - 2z)\hat{j} - 2y\hat{k}$

$\vec{N} = \vec{\nabla} F(4,4,2) = \hat{i} - \hat{j} - 8\hat{k}$

TANGENT PLANE IS

$(x-4) - (y-4) - 8(z-2) = 0$

OR

$x - y - 8z = -16$

3. (4 points) A team of researchers is mapping the ocean floor. Having set up a coordinate system, the depth of the ocean in their vicinity is modeled by the formula

$$D = 250 + 30x^2 + 50\sin\frac{\pi y}{2}; \quad 0 \le x \le 2,\ 0 \le y \le 2,$$

where $D$ is measured in meters, and $x$ and $y$ are measured in kilometers. If the ship is at the location where $x = 1$ and $y = 0.5$, in what direction is the ocean floor steepest?

$$\vec{\nabla} D(x,y) = 60x\,\hat{i} + 25\pi\cos\frac{\pi y}{2}\,\hat{j}$$

$$\vec{\nabla} D(1, 0.5) = 60\,\hat{i} + \frac{25\pi\sqrt{2}}{2}\,\hat{j} \approx \boxed{60\,\hat{i} + 55.536\,\hat{j}}$$

OR

$$\frac{\vec{\nabla} D(1,0.5)}{\|\vec{\nabla} D(1,0.5)\|} \approx \boxed{0.734\,\hat{i} + 0.679\,\hat{j}}$$

4. (6 points) Find the linearization of $f(x, y) = x^2 y \sin(\pi xy)$ at the point $(1, 1)$. Then use your linearization to approximate $f(0.96, 1.02)$.

$$f_x(x,y) = 2xy\sin(\pi xy) + \pi x^2 y^2 \cos(\pi xy)$$

$$f_y(x,y) = x^2 \sin(\pi xy) + \pi x^3 y \cos(\pi xy)$$

$$f_x(1,1) = -\pi$$

$$f_y(1,1) = -\pi \quad \Rightarrow \quad \boxed{L(x,y) = 0 - \pi(x-1) - \pi(y-1) = -\pi x - \pi y + 2\pi}$$

$$f(1,1) = 0$$

$$\boxed{f(0.96, 1.02) \approx L(0.96, 1.02) = 0.0628}$$

5. (8 points) Find and classify the critical points of $f(x,y) = (x+y)(xy+xy^2)$.

$$f(x,y) = x^2y + x^2y^2 + xy^2 + xy^3$$

$$f_x(x,y) = 2xy + 2xy^2 + y^2 + y^3 = 2x(y+y^2) + y(y+y^2)$$

$$= (2x+y)(y+y^2) = (2x+y)(y)(1+y)$$

$$f_y(x,y) = x^2 + 2x^2y + 2xy + 3xy^2$$

$$f_x(x,y) = 0 \Rightarrow y=0,\ y=-1,\ \text{or}\ y=-2x$$

$\underline{y=0}$

$f_y(x,y) = x^2 = 0$

$\Downarrow$

$x = 0$

$\underline{y=-1}$

$f_y(x,y) = x^2 - 2x^2 - 2x + 3x$

$= x - x^2 = x(1-x) = 0$

$\Downarrow$

$x=0$ or $x=1$

$\underline{y=-2x}$

$f_y(x,y) = x^2 - 4x^3 - 4x^2 + 12x^3$

$= 8x^3 - 3x^2 = x^2(8x-3) = 0$

$x=0$ OR $x = \frac{3}{8}$

CRIT PTS ARE

$(0,0),\ (0,-1),\ (1,-1);\ \left(\frac{3}{8}, -\frac{3}{4}\right)$

$$D(x,y) = \begin{vmatrix} 2y + 2y^2 & 2x + 4xy + 2y + 3y^2 \\ 2x + 4xy + 2y + 3y^2 & 2x^2 + 2x + 6xy \end{vmatrix}$$

$$= (2y+2y^2)(2x^2+2x+6xy) - (2x+4xy+2y+3y^2)^2$$

$D(0,0) = 0 \Rightarrow$ 2ND PARTIALS TEST IS INCONCLUSIVE.

TRY SOMETHING ELSE...

LET $\epsilon$ BE A VERY SMALL POS #

$$\left.\begin{array}{l} f(\epsilon,\epsilon) = 2\epsilon^3 + 2\epsilon^4 > 0 \\ f(-\epsilon,-\epsilon) = -2\epsilon^3 + 2\epsilon^4 < 0 \end{array}\right\} \text{POS AND NEG VALUES AROUND } (0,0)$$

$\Downarrow$

$\boxed{(0,0,0) \text{ IS A SADDLE PT.}}$

$D(0,-1) = -1 < 0 \Rightarrow \boxed{(0,-1,0) \text{ IS A SADDLE PT.}}$

$D(1,-1) = -1 < 0 \Rightarrow \boxed{(1,-1,0) \text{ IS A SADDLE PT.}}$

$D\left(\frac{3}{8}, -\frac{3}{4}\right) = \frac{27}{128} > 0$, $f_{xx}\left(\frac{3}{8}, -\frac{3}{4}\right) = -\frac{3}{8}$

$\Rightarrow \boxed{f\left(\frac{3}{8}, -\frac{3}{4}\right) = \frac{27}{1024} \text{ IS A REL MAX}}$

6. (12 points)

(a) Use Lagrange multipliers to find the point on the plane $2x + y + 3z = 7$ that is closest to $(1, 1, 1)$.

$$D(x,y,z) = \sqrt{(x-1)^2+(y-1)^2+(z-1)^2}$$

WE'LL MINIMIZE $D^2 = (x-1)^2+(y-1)^2+(z-1)^2$

s.t. $2x+y+3z=7$

$$2(x-1) = 2\lambda \Rightarrow x = \frac{2\lambda+2}{2}$$
$$2(y-1) = \lambda \Rightarrow y = \frac{\lambda+2}{2}$$
$$2(z-1) = 3\lambda \Rightarrow z = \frac{3\lambda+2}{2}$$

$$2\left(\frac{2\lambda+2}{2}\right) + \frac{\lambda+2}{2} + 3\left(\frac{3\lambda+2}{2}\right) = 7$$

$$4\lambda+4+\lambda+2+9\lambda+6 = 14$$

$$14\lambda = 2$$
$$\lambda = \frac{1}{7}$$

$$x = \frac{8}{7},\ y = \frac{15}{14},\ z = \frac{17}{14}$$

$\boxed{\left(\frac{8}{7}, \frac{15}{14}, \frac{17}{14}\right)}$ MUST GIVE A MIN DISTANCE -- THERE CANNOT BE A MAX.

(b) Find the distance from your solution in part (a) to the point $(1, 1, 1)$.

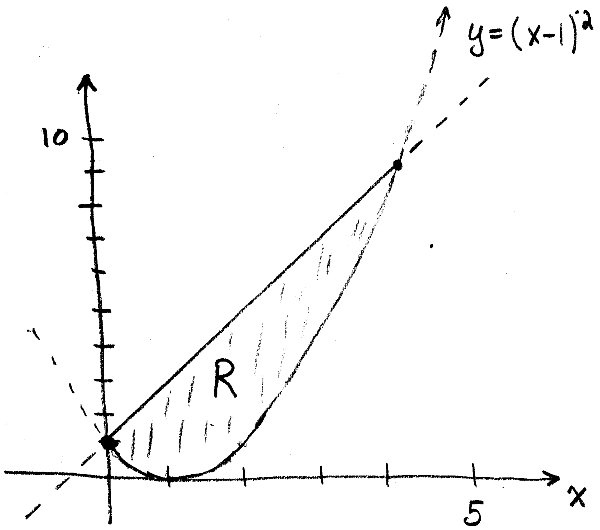
$$D\left(\frac{8}{7}, \frac{15}{14}, \frac{17}{14}\right) = \sqrt{\left(\frac{1}{7}\right)^2+\left(\frac{1}{14}\right)^2+\left(\frac{3}{14}\right)^2}$$

$$= \sqrt{\frac{1}{14}} = \boxed{\frac{1}{\sqrt{14}} \approx 0.267}$$

(c) Use the techniques of section 11.5 to find the distance from the plane to the point $(1, 1, 1)$.

$$D = \frac{|2(1)+(1)+3(1)-7|}{\sqrt{2^2+1^2+3^2}} = \frac{|6-7|}{\sqrt{14}} = \boxed{\frac{1}{\sqrt{14}}}$$

7. (10 points) Evaluate the double integral $\iint\limits_R (3xy + 6x^2)\,dA$, where $R$ is the first quadrant region bounded by the graphs of $y = (x-1)^2$ and $y = 2x + 1$.




$$(x-1)^2 = 2x+1$$
$$x^2 - 2x + 1 = 2x + 1$$
$$x^2 - 4x = 0$$
$$x(x-4) = 0$$
$$x = 0,\ x = 4$$

$$\iint\limits_R (3xy + 6x^2)\,dA = \int_{x=0}^{x=4} \int_{y=(x-1)^2}^{y=2x+1} (3xy + 6x^2)\,dy\,dx$$

$$= \int_0^4 \frac{3}{2}xy^2 + 6x^2y \Big|_{(x-1)^2}^{2x+1} dx = \int_0^4 \Big[\frac{3}{2}x(2x+1)^2 + 6x^2(2x+1) - \frac{3}{2}x(x-1)^4 - 6x^2(x-1)^2\Big]dx$$

$$= \int_0^4 6x^3 + 6x^2 + \cancel{\frac{3}{2}x} + 12x^3 + 6x^2 - \frac{3}{2}x^5 + \cancel{6x^4} - 9x^3 + \cancel{6x^2} - \cancel{\frac{3}{2}x} - \cancel{6x^4} + 12x^3 - \cancel{6x^2}\,dx$$

$$= \int_0^4 \left(-\frac{3}{2}x^5 + 21x^3 + 12x^2\right)dx = -\frac{1}{4}x^6 + \frac{21}{4}x^4 + 4x^3 \Big|_0^4$$

$$= -1024 + 1344 + 256$$

$$= \boxed{576}$$