Math 173 - Test 1
February 20, 2014

Name: *key*

Score ________

Show all work to receive full credit. Supply explanations where necessary.

1. (6 points) Find the component form of the vector of magnitude 3 that has the direction from $P(5, 2, -1)$ to $Q(3, -3, 4)$.

$$\vec{PQ} = (3-5)\hat{i} + (-3-2)\hat{j} + (4-(-1))\hat{k}$$
$$= -2\hat{i} - 5\hat{j} + 5\hat{k}$$

$$\|\vec{PQ}\| = \sqrt{(-2)^2 + (-5)^2 + 5^2}$$
$$= \sqrt{4 + 25 + 25} = \sqrt{54}$$

$$\boxed{\frac{3}{\sqrt{54}}\left(-2\hat{i} - 5\hat{j} + 5\hat{k}\right)}$$

2. (8 points) Find the angle between the planes described by the following equations.

$$2x - y + 2z = 7 \qquad -5x + 3z = 12$$

$$\vec{N}_1 = 2\hat{i} - \hat{j} + 2\hat{k} \qquad \vec{N}_2 = -5\hat{i} + 3\hat{k}$$

$$\|\vec{N}_1\| = \sqrt{4+1+4} = 3 \qquad \|\vec{N}_2\| = \sqrt{25+9} = \sqrt{34}$$

$$\cos\theta = \frac{\vec{N}_1 \cdot \vec{N}_2}{\|\vec{N}_1\|\,\|\vec{N}_2\|} = \frac{-10+6}{3\sqrt{34}} = \frac{-4}{3\sqrt{34}}$$

$\theta = \cos^{-1}\left(\frac{-4}{3\sqrt{34}}\right) \approx 103.22°$ OR ITS SUPPLEMENT $\boxed{76.78°}$

3. (6 points) What does it mean for two vectors to be orthogonal? Find a nonzero vector orthogonal to $\vec{u} = \sqrt{2}\hat{\imath} - 5\hat{\jmath} + \frac{3}{2}\hat{k}$.

TWO VECTORS ARE ORTHOGONAL IF THEIR DOT PRODUCT IS ZERO.

FOR EXAMPLE, $\boxed{\vec{v} = 3\hat{j} + 10\hat{k}}$ IS ORTHOG TO $\vec{u}$

BECAUSE $\vec{u}\cdot\vec{v} = (\sqrt{2})(0) + (-5)(3) + \left(\frac{3}{2}\right)(10)$
$= 0$

4. (8 points) The vector $\vec{u}$ lies in the xy-plane, has magnitude 4, and makes an angle of $120°$ with the positive x-axis. The vector $\vec{w}$ is given by $\vec{w} = \hat{\imath} + \hat{\jmath}$. Compute $\|\vec{u} + \vec{w}\|$.

$$\vec{u} = 4\cos 120°\,\hat{\imath} + 4\sin 120°\,\hat{\jmath} = 4\left(-\tfrac{1}{2}\right)\hat{\imath} + 4\left(\tfrac{\sqrt{3}}{2}\right)\hat{\jmath}$$

$$= -2\hat{\imath} + 2\sqrt{3}\,\hat{\jmath}$$

$$\vec{u} + \vec{w} = -\hat{\imath} + (2\sqrt{3}+1)\hat{\jmath}$$

$$\|\vec{u} + \vec{w}\| = \sqrt{(-1)^2 + (2\sqrt{3}+1)^2} = \sqrt{20.928\ldots}$$

$$\approx \boxed{4.57}$$

5. (8 points) Find the midpoint of the line segment connecting the points $(1, 4, -7)$ and $(5, 2, -3)$. Using that point as your initial point, find a set of parametric equations for the line through the points.

$P(1,4,-7) \qquad Q(5,2,-3)$

MIDPOINT IS $\left(\dfrac{1+5}{2}, \dfrac{4+2}{2}, \dfrac{-7+(-3)}{2}\right) = (3, 3, -5)$

$$\vec{PQ} = (5-1)\hat{\imath} + (2-4)\hat{\jmath} + (-3-(-7))\hat{k}$$

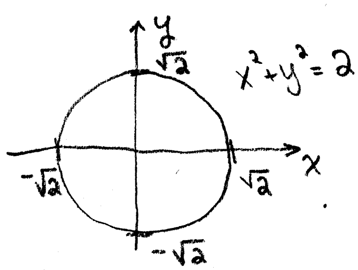
$$= 4\hat{\imath} - 2\hat{\jmath} + 4\hat{k}$$

LINE IS GIVEN BY

$$x = 3 + 4t$$
$$y = 3 - 2t$$
$$z = -5 + 4t$$

6. (10 points) Consider the surface described by the equation $2x^2 + 2y^2 - z^2 = 4$.

(a) Describe (or sketch) in detail the level curve obtained by fixing $z = 0$.

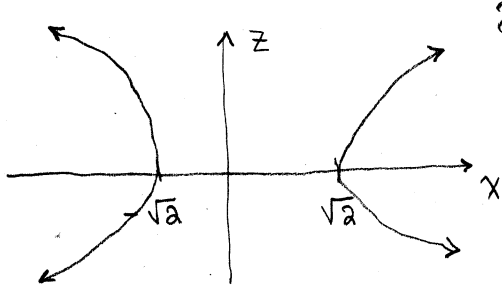


$2x^2 + 2y^2 = 4$

$x^2 + y^2 = 2$

THIS IS THE CIRCLE CENTERED AT $(0,0)$ WITH RADIUS $\sqrt{2}$.

(b) Describe (or sketch) in detail the level curve obtained by fixing $y = 0$.



$2x^2 - z^2 = 4$

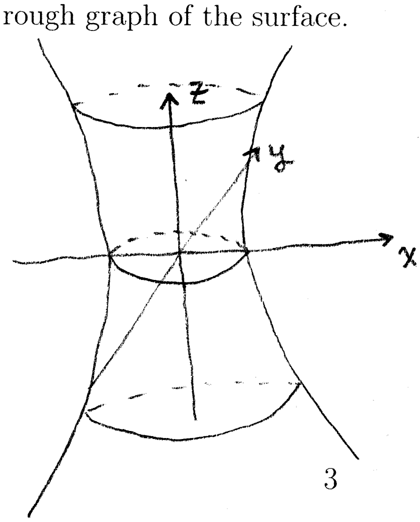
IN XZ-PLANE

HYPERBOLA WITH "CENTER" $(0,0)$ AND VERTICES $(\pm\sqrt{2}, 0)$

(c) Identify the surface.

HYPERBOLOID OF ONE SHEET

(d) Sketch a rough graph of the surface.



7. (8 points) Find a unit vector in the xy-plane normal to the graph of $x^3 + y^3 + xy = 5$ at the point where $(x, y) = (2, -1)$.

$$\frac{d}{dx}\left(x^3+y^3+xy\right) = \frac{d}{dx}(5)$$

$$3x^2 + 3y^2\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$$

$$\left(3y^2+x\right)\frac{dy}{dx} = -3x^2-y$$

$$\frac{dy}{dx} = \frac{-3x^2-y}{3y^2+x}$$

$$\left.\frac{dy}{dx}\right|_{(2,-1)} = \frac{-12+1}{3+2} = -\frac{11}{5}$$

NORMAL VECTOR HAS SLOPE $\frac{5}{11}$

$11\hat{i} + 5\hat{j}$

$\sqrt{121+25} = \sqrt{146}$

$$\boxed{\frac{1}{\sqrt{146}}\left(11\hat{i}+5\hat{j}\right)}$$

8. (6 points) Find numbers a and b so that $\vec{u} = 5\hat{\imath} - 3\hat{\jmath} + a\hat{k}$ and $\vec{v} = b\hat{\imath} + 6\hat{\jmath} - 4\hat{k}$ are parallel. Do the resulting vectors point in the same direction or opposite directions?

$$5 = bt$$
$$-3 = 6t \rightarrow t = -\frac{1}{2}$$
$$a = -4t$$

$$\Rightarrow \boxed{\begin{array}{l} b = -10 \\ a = 2 \end{array}}$$

$$\vec{u} = -\frac{1}{2}\vec{v}$$

OPPOSITE DIRECTIONS

9. (8 points) The following vectors are orthogonal:

$$\vec{v} = \hat{\imath} + 3\hat{\jmath} - 2\hat{k} \qquad \vec{w} = -5\hat{\imath} + \hat{\jmath} - \hat{k}$$

Compute $\|\vec{v} \times \vec{w}\|$.

SINCE $\vec{v}$ & $\vec{w}$ ARE ORTHOGONAL,

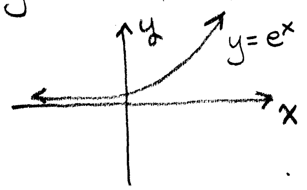
$$\|\vec{v}\times\vec{w}\| = \|\vec{v}\|\,\|\vec{w}\| = \sqrt{1+9+4}\,\sqrt{25+1+1}$$

$$= \sqrt{14}\,\sqrt{27} = \boxed{\sqrt{378} \approx 19.44}$$

10. (6 points) Describe (or sketch) the 3D surface whose equation is given by $y = e^x$.

THE 3D graph OF $y = e^x$ IS A CYLINDER MADE UP OF ALL LINES PARALLEL TO THE Z-AXIS AND PASSING THROUGH THE PLANE CURVE $y = e^x$.

· IMAGINE THE CURVE



COMING IN AND OUT THE PLANE OF THE PAPER.

11. (8 points) Find the area of the triangle with vertices $(9, 1, -2)$, $(1, 1, 1)$, and $(-3, 4, 2)$.

P Q R

$\vec{PQ} = -8\hat{i} + 3\hat{k}$

$\vec{PR} = -12\hat{i} + 3\hat{j} + 4\hat{k}$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -8 & 0 & 3 \\ -12 & 3 & 4 \end{vmatrix}$$

$$= \hat{i}(-9) - \hat{j}(4) + \hat{k}(-24) = -9\hat{i} - 4\hat{j} - 24\hat{k}$$

$\text{AREA} = \frac{1}{2}\|\vec{PQ} \times \vec{PR}\|$

$= \frac{1}{2}\sqrt{81 + 16 + 576}$

$= \frac{\sqrt{673}}{2} \approx 12.97$

12. (8 points) Find an equation of the plane parallel to $x - 8y + 7z = 15$ but passing through the point $(1, 2, 3)$.

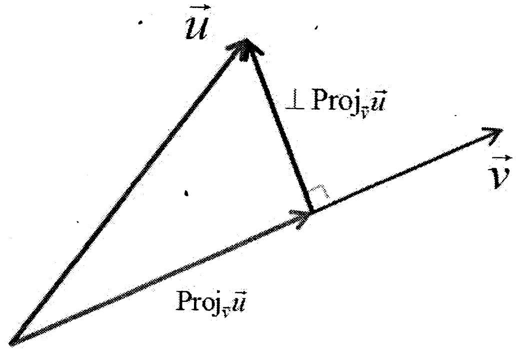
PARALLEL MEANS SAME COEFFICIENTS!

$x - 8y + 7z = d$

$(1) - 8(2) + 7(3) = 1 - 16 + 21 = 6$

$$\boxed{x - 8y + 7z = 6}$$

13. (10 points) In the figure below, $\vec{u} = \hat{\imath} - \hat{\jmath} - \hat{k}$ and $\vec{v} = 2\hat{\imath} + \hat{\jmath} - 4\hat{k}$. Determine the projection, $\text{Proj}_{\vec{v}}\vec{u}$, and the vector labeled $\perp \text{Proj}_{\vec{v}}\vec{u}$.



$\vec{u}\cdot\vec{v} = (1)(2) + (-1)(1) + (-1)(-4) = 5$

$\vec{v}\cdot\vec{v} = 4 + 1 + 16 = 21$

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{5}{21}(2\hat{\imath} + \hat{\jmath} - 4\hat{k}) = \frac{10}{21}\hat{\imath} + \frac{5}{21}\hat{\jmath} - \frac{20}{21}\hat{k}$$

$$\perp \text{proj}_{\vec{v}}\vec{u} = \vec{u} - \text{proj}_{\vec{v}}\vec{u} = \frac{11}{21}\hat{\imath} - \frac{26}{21}\hat{\jmath} - \frac{1}{21}\hat{k}$$

14. (5 points ex credit) A line segment connects the points $P(2,3,1)$ and $Q(5,-4,2)$. Find the coordinates of the point on the line segment that lies two-thirds of the way along the segment in the direction of $\vec{PQ}$.

$\vec{PQ} = 3\hat{\imath} - 7\hat{\jmath} + \hat{k}$

Using $P(2,3,1)$

PARAMETRIC EQUATIONS ARE

$x = 2 + 3t$

$y = 3 - 7t$ $\quad 0 \le t \le 1$

$z = 1 + t$

Now let $t = \frac{2}{3}$ to get

$(x,y,z) = \boxed{\left(4, -\frac{5}{3}, \frac{5}{3}\right)}$