Math 173 - 1st Final Exam

May 5, 2011

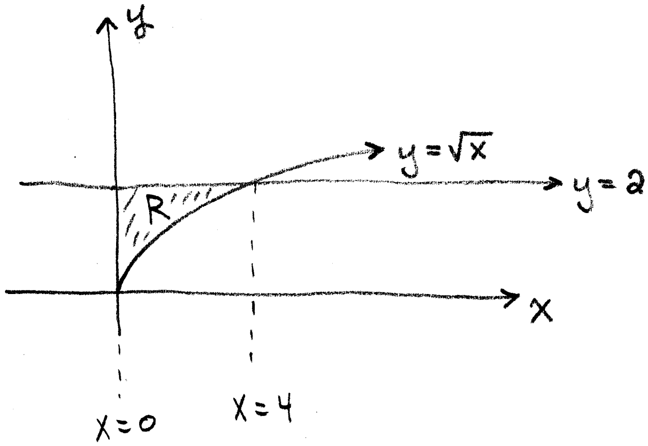
Name key

Score

Show all work. Supply explanations when necessary. **Unless otherwise specified, you may use your calculator to evaluate any integrals.** Each problem is worth 10 points.

1. Sketch the region of integration, reverse the order of integration, and **evaluate the iterated integral by hand.**

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{x}{y^5+1}\, dy\, dx$$



$$\int_{y=0}^{y=2} \int_{x=0}^{x=y^2} \frac{x}{y^5+1}\, dx\, dy$$

$$= \frac{1}{2}\int_0^2 \frac{y^4}{y^5+1}\, dy \qquad u = y^5+1 \quad du = 5y^4 dy$$

$$= \frac{1}{10}\int_1^{33} \frac{1}{u}\, du = \boxed{\frac{1}{10}\ln(33)}$$

2. Find and classify all relative extreme values of the function $f(x,y)$.

$$f(x,y) = x^2 + xy + 2y^2 + x - 3y + 10$$

$f_x(x,y) = 2x + y + 1 = 0$

$f_y(x,y) = x + 4y - 3 = 0$

$2x + y = -1$

$2x + 8y = 6$

$7y = 7 \Rightarrow y = 1,\ x = -1$

ONLY CRIT PT IS $(-1,1)$

$$d(x,y) = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 7$$

$d(-1,1) = 7 > 0$ AND $f_{xx}(-1,1) = 2 > 0 \Rightarrow$ $\boxed{f(-1,1) = 8 \text{ IS A RELATIVE MIN}}$

3. A plane passes through the points $P\,(2,1,3)$, $Q\,(-7,6,-1)$ and $R\,(3,0,-1)$. Find a set of parametric equations for the line normal to the plane and passing through $(2,1,3)$.

$\vec{PR} = \hat{i} - \hat{j} - 4\hat{k}$

$\vec{QR} = 10\hat{i} - 6\hat{j}$

$$\vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -4 \\ 10 & -6 & 0 \end{vmatrix} = -24\hat{i} - 40\hat{j} + 4\hat{k}$$

I'LL USE $-6\hat{i} - 10\hat{j} + \hat{k}$

AND $P\,(2,1,3)$

$$\boxed{\begin{aligned} x &= -6t + 2 \\ y &= -10t + 1 \\ z &= t + 3 \end{aligned}}$$

4. Find a vector of length 3 that is normal to the plane given by $5x - 3y + 2z = 10$.

$\vec{N} = 5\hat{i} - 3\hat{j} + 2\hat{k}$

$|\vec{N}| = \sqrt{25 + 9 + 4} = \sqrt{38}$

$$\boxed{\frac{3}{\sqrt{38}}\left(5\hat{i} - 3\hat{j} + 2\hat{k}\right)}$$

5. Let $\vec{r}(t) = t^2\hat{\imath} + (\sin t - t\cos t)\hat{\jmath} + (\cos t + t\sin t)\hat{k}$. Find the unit tangent vector and the principal unit normal vector. (Hint: If you're doing everything correctly, the computations should not be messy.)

$$\vec{r}\,'(t) = 2t\hat{\imath} + (\cos t - \cos t + t\sin t)\hat{\jmath} + (-\sin t + \sin t + t\cos t)\hat{k}$$

$$= 2t\hat{\imath} + t\sin t\hat{\jmath} + t\cos t\hat{k}$$

$$|\vec{r}\,'(t)| = \sqrt{4t^2 + t^2} = \sqrt{5}\,t, \text{ Assuming } t > 0$$

$$\boxed{\hat{T}(t) = \frac{1}{\sqrt{5}}(2\hat{\imath} + \sin t\hat{\jmath} + \cos t\hat{k})}$$

$$\hat{T}'(t) = \frac{1}{\sqrt{5}}(\cos t\hat{\jmath} - \sin t\hat{k})$$

$$\boxed{\hat{N}(t) = \cos t\hat{\jmath} - \sin t\hat{k}}$$

$$|\hat{T}'(t)| = \frac{1}{\sqrt{5}}$$

6. Let $\vec{a} = 5\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ and $\vec{c} = \hat{\imath} - 4\hat{\jmath} + 2\hat{k}$. Now let $\vec{g} = \text{proj}_{\vec{c}}\vec{a}$. Compute $\vec{g}$ and then compute $\vec{g} + \vec{c}$.

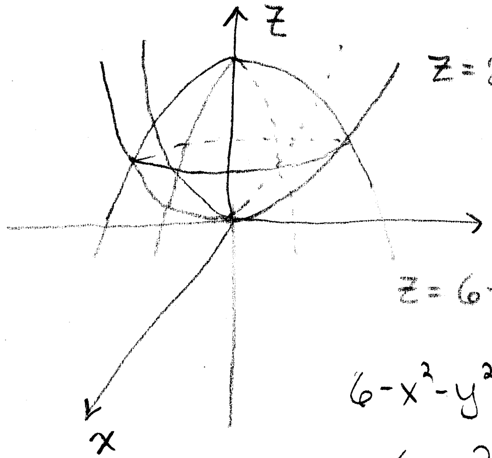
$$\text{proj}_{\vec{c}}\,\vec{a} = \frac{\vec{a}\cdot\vec{c}}{\vec{c}\cdot\vec{c}}\vec{c} = \frac{5-12-4}{1+16+4} = -\frac{11}{21}(\hat{\imath} - 4\hat{\jmath} + 2\hat{k})$$

$$\vec{g} = \boxed{-\frac{11}{21}\hat{\imath} + \frac{44}{21}\hat{\jmath} - \frac{22}{31}\hat{k}}$$

$$\vec{g} + \vec{c} = -\frac{11}{21}\vec{c} + \vec{c} = \frac{10}{21}\vec{c} = \boxed{\frac{10}{21}\hat{\imath} - \frac{40}{21}\hat{\jmath} + \frac{20}{21}\hat{k}}$$

7. Let E be the space region bounded by the paraboloids $z = 2x^2 + 2y^2$ and $z = 6 - x^2 - y^2$. Evaluate the following triple integral. (Hint: It's easier in cylidrical coordinates.)

$$\iiint_E (x^2 + y^2)\, dV$$



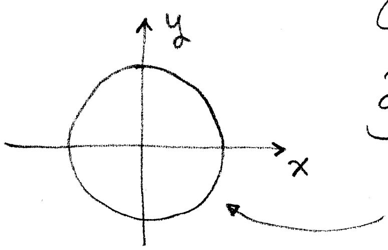
$z = 2x^2 + 2y^2$

$z = 6 - x^2 - y^2$

$$6 - x^2 - y^2 = 2x^2 + 2y^2$$

$$6 = 3x^2 + 3y^2$$

$$\underbrace{2 = x^2 + y^2}_{r = \sqrt{2}}$$



$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\sqrt{2}} \int_{z=2r^2}^{z=6-r^2} r^2 \cdot r \, dz\, dr\, d\theta$$

$$= \boxed{4\pi}$$

From TI-89

8. A batter hits a baseball 3 ft above the ground toward the center field fence, which is 10 ft high and 400 ft from home plate. The ball leaves the bat with a speed of 115 ft/sec at an angle of 50° above the horizontal. Does the ball clear the fence? (Use $g = 32\,\text{ft/sec}^2$.)

$$\vec{r}(t) = (v_0 \cos\theta\, t + x_0)\,\hat{i} + \left(-\tfrac{1}{2} g t^2 + v_0 \sin\theta\, t + y_0\right)\hat{j}$$

$$\vec{r}(t) = (115 \cos 50^\circ)\, t\, \hat{i} + \left(-16t^2 + 115 \sin 50^\circ t + 3\right)\hat{j}$$

$$115 \cos 50^\circ t = 400$$

$$t = \frac{400}{115 \cos 50^\circ} \approx 5.411 \text{ sec}$$

$$-16(5.411)^2 + 115 \sin 50^\circ (5.411) + 3 \approx 11.202 \text{ FT} > 10 \text{ FT}$$

YES, THE BALL WILL CLEAR THE FENCE.

9. Use Lagrange multipliers to find the extreme values of $f(x,y) = \frac{1}{3}x^3 + y^2$ on the unit circle $x^2 + y^2 = 1$.

$f(x,y) = \frac{1}{3}x^3 + y^2$

$g(x,y) = x^2 + y^2$

$\vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow x^2 = \lambda 2x$

$2y = \lambda 2y \Longrightarrow 2y - \lambda 2y = 0$

$g(x,y) = 1 \Rightarrow x^2 + y^2 = 1$

$2y(1-\lambda) = 0$

$y = 0$ OR $\lambda = 1$

$y=0$: $x = \pm 1$

$\lambda = 1$: $x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0,\ x = 2$

$x=0$: $y = \pm 1$

$x=2$: NOT POSSIBLE

CRITICAL POINTS ARE $(1,0),\ (-1,0),\ (0,1);\ (0,-1)$

$f(1,0) = \frac{1}{3}$

$\boxed{f(-1,0) = -\frac{1}{3}}$ ← MIN VALUE

$f(0,-1) = 1$

$f(0,1) = 1$ ← MAX VALUE

10. The curves described by the vector-valued functions $\vec{r_1}(t) = t\hat{\imath} + (t^2+3)\hat{\jmath} + (6-5t)\hat{k}$ and $\vec{r_2}(t) = t^2\hat{\imath} + (2t+2)\hat{\jmath} + t^3\hat{k}$ intersect at the point $(1,4,1)$. Find the angle of intersection of the curves. (Hint: Find the angle between the velocity vectors at the point of intersection.)

$t\hat{\imath} + (t^2+3)\hat{\jmath} + (6-5t)\hat{k} = \hat{\imath} + 4\hat{\jmath} + \hat{k}$

$\Longrightarrow t = 1$

$\vec{r}_1'(t) = \hat{\imath} + 2t\hat{\jmath} - 5\hat{k}$ $\qquad \vec{r}_1'(1) = \hat{\imath} + 2\hat{\jmath} - 5\hat{k}$

$\vec{r}_2'(t) = 2t\hat{\imath} + 2\hat{\jmath} + 3t^2\hat{k}$ $\qquad \vec{r}_2'(1) = 2\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

} FIND ∠ BETWEEN.

$$\cos\theta = \frac{2+4-15}{\sqrt{30}\sqrt{17}} \approx \frac{-9}{\sqrt{510}}$$

$$\theta = \cos^{-1}\left(\frac{-9}{\sqrt{510}}\right) \approx \underline{113.486^\circ}$$

11. The temperature at a point (x, y) is given by

$$T(x,y) = \frac{xy}{x^2+y^2+1}$$

where T is measured in °C and x and y in meters.

(a) At the point $(1,1)$, in what direction does the temperature **decrease** the fastest?

OPPOSITE $\vec{\nabla} T(1,1)$.

$$\vec{\nabla} T(x,y) = \frac{y(x^2+y^2+1) - xy(2x)}{(x^2+y^2+1)^2}\,\hat{\imath} + \frac{x(x^2+y^2+1) - xy(2y)}{(x^2+y^2+1)^2}\,\hat{\jmath}$$

$$-\vec{\nabla} T(1,1) = -\left(\frac{1}{9}\hat{\imath} + \frac{1}{9}\hat{\jmath}\right)$$

$$\boxed{-\frac{1}{9}\hat{\imath} - \frac{1}{9}\hat{\jmath}} \quad \text{OR} \quad \boxed{-\hat{\imath} - \hat{\jmath}}$$

(b) Find the rate of change of temperature at the $\underbrace{\text{point } (1,1)}_{P}$ in the direction toward the $\underbrace{\text{point } (2,3)}_{Q}$.

$$\vec{PQ} = \hat{\imath} + 2\hat{\jmath}$$

$$\frac{\vec{\nabla} T(1,1) \cdot [\hat{\imath} + 2\hat{\jmath}]}{\sqrt{1+4}} = \frac{\frac{1}{9} + \frac{2}{9}}{\sqrt{5}} = \boxed{\frac{1}{3\sqrt{5}}}$$

12. Show that the limit does not exist. (Hint: $y = x^2$ is a path through $(0,0)$.)

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$$

PATH $x = 0$:

$$\lim_{y\to 0} \frac{0}{y^2} = 0$$

PATH $y = x^2$:

$$\lim_{y\to 0} \frac{y^2}{y^2+y^2} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}$$

$0 \neq \frac{1}{2}$

LIMIT DNE.

13. Set up the definite integral that gives the length of the space curve defined by the following vector-valued function over the interval from $t = 0$ to $t = 5$. Use your calculator to approximate the value of the integral.

$$\vec{r}(t) = \sin(3t)\hat{\imath} + t\cos(t)\hat{\jmath} + e^{-5t}\hat{k}$$

$$\vec{r}\,'(t) = 3\cos 3t\,\hat{\imath} + (\cos t - t\sin t)\hat{\jmath} - 5e^{-5t}\hat{k}$$

$$\int_0^5 \sqrt{9\cos^2 3t + (\cos t - t\sin t)^2 + 25e^{-10t}}\, dt$$

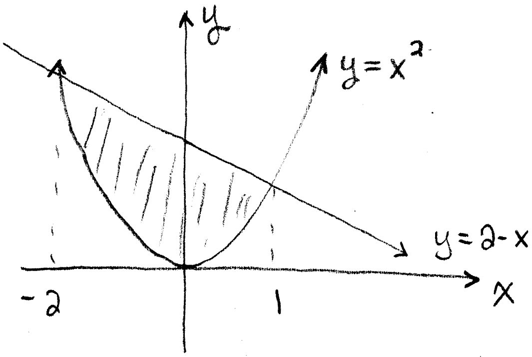
$$\approx 15.76$$

14. Suppose u is a function of x, y, and z, where x, y, and z are functions of r, s, and t. State the chain rule formulas for $\partial u/\partial r$ and $\partial u/\partial t$.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$

15. A thin plate is bounded by the graphs of $y = x^2$ and $y = 2 - x$. The density of the plate at the point (x, y) is given by $\rho(x, y) = 5 + xy$. Find the center of mass of the plate. (Be sure to type $x * y$ on your calculator.)



$$x^2 = 2 - x \Rightarrow x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0$$

$$x = -2, \; x = 1$$

$$\text{MASS} = \int_{x=-2}^{x=1} \int_{y=x^2}^{y=2-x} (5+xy)\,dy\,dx$$

$$= \frac{135}{8}$$

$$M_y = \int_{-2}^{1} \int_{x^2}^{2-x} x(5+xy)\,dy\,dx$$

$$= \frac{-513}{140}$$

$$M_x = \int_{-2}^{1} \int_{x^2}^{2-x} y(5+xy)\,dy\,dx$$

$$= \frac{837}{40}$$

$$\text{CENTER OF MASS} = \left(\frac{M_y}{M}, \frac{M_x}{M} \right)$$

$$= \boxed{\left(-\frac{38}{175}, \frac{31}{25} \right) \approx (-0.217, 1.24)}$$