Math 173 - Test 3a
April 28, 2011

Name key

Score

Show all work. Supply explanations when necessary.

1. (8 points) Find and classify all critical points of the following function.

$$f(x,y) = 2x^2 + 2xy + y^2 + 2x - 3$$

$f_x(x,y) = 4x + 2y + 2 = 0$

$f_y(x,y) = 2x + 2y = 0 \Rightarrow y = -x \Rightarrow 4x - 2x + 2 = 0$

$\Rightarrow 2x + 2 = 0$

$\Rightarrow x = -1$

$\Rightarrow y = 1$

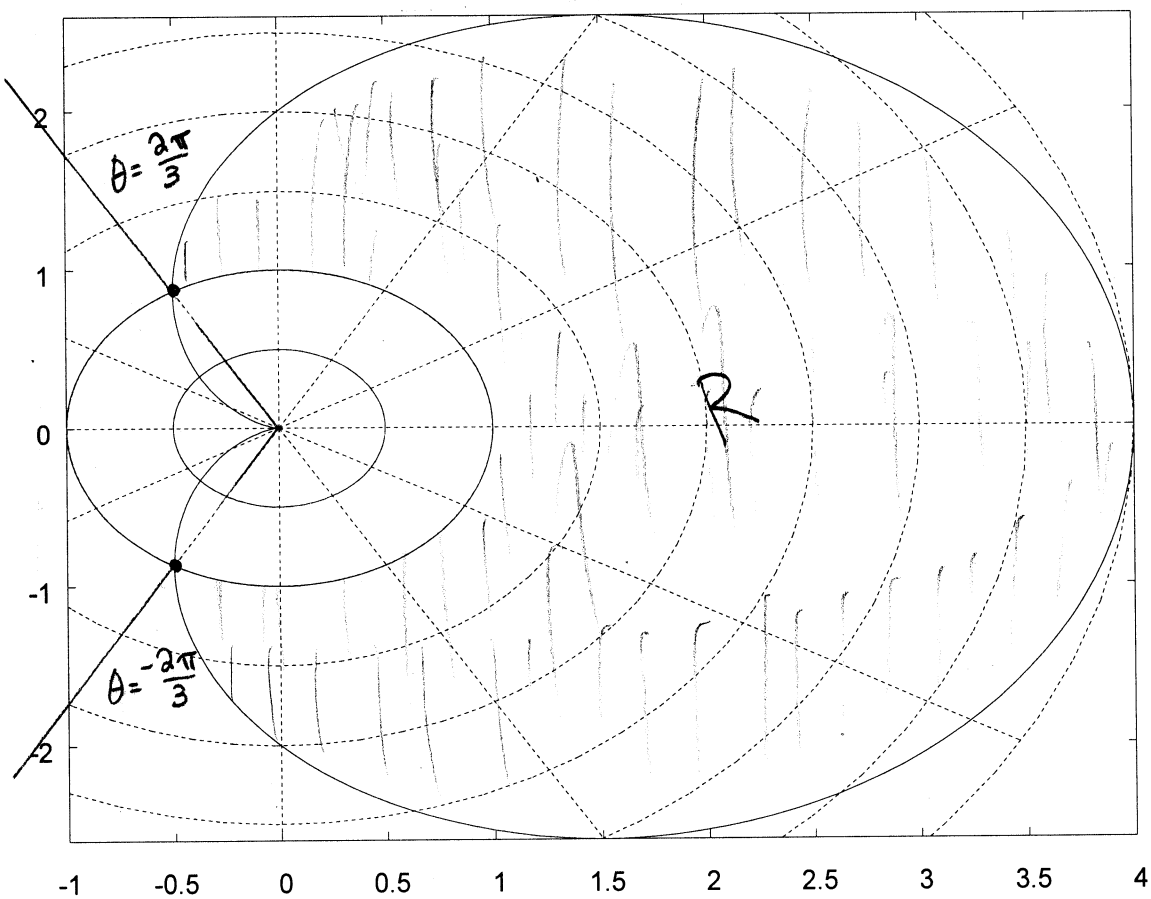
THE ONLY CRITICAL PT IS $(-1,1)$.

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 2 & 2 \end{vmatrix} = 8 - 4 = 4$$

$D(-1,1) = 4 > 0$ AND $f_{xx}(-1,1) = 4 > 0$

$\Rightarrow$ $f(-1,1) = -4$ IS A RELATIVE MINIMUM.

2. (10 points) Let R be the polar region inside the graph of the $r = 2+2\cos\theta$ and outside the graph of $r = 1$. Sketch the region R and then evaluate $\iint_R (2r+3)\, dA$. You may use your calculator to evaluate the iterated integral.



$$1 = 2 + 2\cos\theta \Rightarrow -\frac{1}{2} = \cos\theta \Rightarrow \theta = -\frac{2\pi}{3}, \frac{2\pi}{3}$$

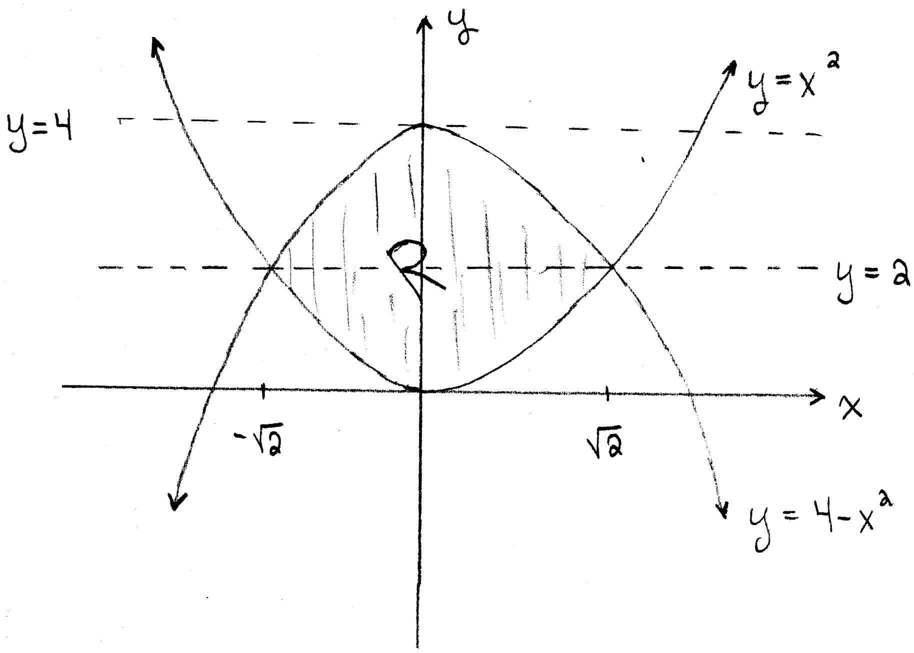
$$\iint_R (2r+3)\, dA = \int_{\theta=-\frac{2\pi}{3}}^{\theta=\frac{2\pi}{3}} \int_{r=1}^{r=2+2\cos\theta} (2r+3)\, r \, dr\, d\theta$$

$$= \frac{477\sqrt{3} + 484\pi}{18}$$

$$\approx 130.37$$

3. (7 points) Sketch the region of integration and then write the definite integral with the reversed order of integration. Two integrals will be required. Do not evaluate.

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{x^2}^{4-x^2} (e^{4x} + 2y - 8)\, dy\, dx$$



$x^2 = 4 - x^2$

$\Rightarrow 2x^2 = 4$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm\sqrt{2}$

$\Rightarrow y = 2$

$$\int_{y=0}^{y=2} \int_{x=-\sqrt{y}}^{x=\sqrt{y}} (e^{4x} + 2y - 8)\, dx\, dy$$

$$+ \int_{y=2}^{y=4} \int_{x=-\sqrt{4-y}}^{x=\sqrt{4-y}} (e^{4x} + 2y - 8)\, dx\, dy$$

4. (6 points) Evaluate the iterated integral by hand. Show all work, but you may use your calculator to check.

$$\int_0^1 \int_0^{\sqrt{1-y^2}} (x+y)\,dx\,dy$$

$$\int_0^1 \left(\frac{1}{2}x^2 + yx\right)\Big|_{x=0}^{x=\sqrt{1-y^2}} dy = \int_0^1 \left[\frac{1}{2}(1-y^2) + y\sqrt{1-y^2}\right] dy$$

$$= \frac{1}{2}\int_0^1 (1-y^2)\,dy + \int_0^1 y\sqrt{1-y^2}\,dy$$

$u = 1-y^2$
$du = -2y\,dy$

$$-\frac{1}{2}\int_1^0 u^{1/2}\,du = \frac{1}{2}\int_0^1 u^{1/2}\,du$$

$$= \frac{1}{3}u^{3/2}\Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{6} + \frac{1}{3} = \boxed{\frac{2}{3}}$$

5. (5 points) One critical point of the function

$$f(x,y) = y^3 - 3x^2y - 3y^2 - 3x^2 + 1$$

is $(0,2)$. Use the 2nd partials test to classify this critical point.

$f_x(x,y) = -6xy - 6x$

$f_y(x,y) = 3y^2 - 3x^2 - 6y$

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

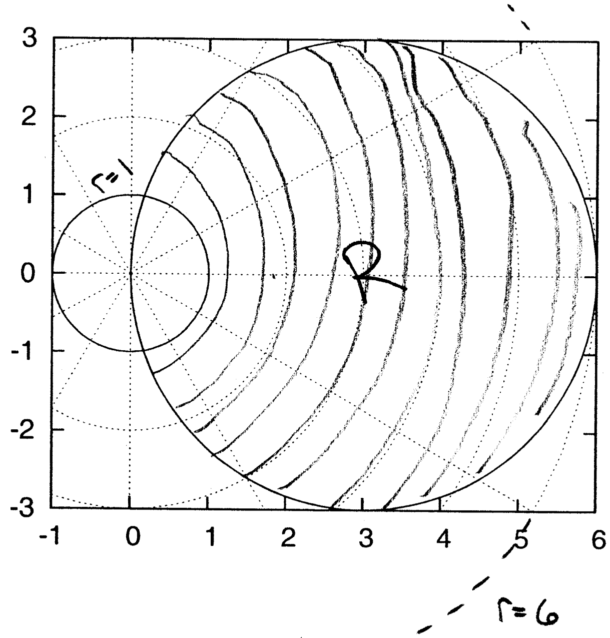
$$= \begin{vmatrix} -6y-6 & -6x \\ -6x & 6y-6 \end{vmatrix} = -(6y+6)(6y-6) - 36x^2$$

$$= -36y^2 + 36 - 36x^2$$

$$D(0,2) = -36(4) + 36 = -108 < 0$$

$\boxed{(0,2,-3) \text{ IS A SADDLE PT.}}$

6. (7 points) Let R be the polar region inside the graph of $r = 6\cos\theta$ and outside the graph of $r = 1$. Set up, but do not evaluate, the iterated integral in the $d\theta\, dr$ order that gives the area of R.



$$\int_{r=1}^{r=6} \int_{\theta=-\cos^{-1}\left(\frac{r}{6}\right)}^{\theta=\cos^{-1}\left(\frac{r}{6}\right)} r \, d\theta\, dr$$

OR $$2\int_{r=1}^{r=6} \int_{\theta=0}^{\theta=\cos^{-1}\left(\frac{r}{6}\right)} r \, d\theta\, dr$$

$$= 17\cos^{-1}\left(\frac{1}{6}\right) + \frac{\sqrt{35}}{2}$$

7. (7 points) Use Lagrange multipliers to find the minimum value of

$$f(x, y, z) = 2x^2 + y^2 + 3z^2$$

subject to the constraint $2x - 3y - 4z = 49$.

$\hat{i}: \quad 4x = \lambda(2) \Rightarrow x = \frac{\lambda}{2}$

$\hat{j}: \quad 2y = \lambda(-3) \Rightarrow y = -\frac{3\lambda}{2}$

$\hat{k}: \quad 6z = \lambda(-4) \Rightarrow z = -\frac{2\lambda}{3}$

$2x - 3y - 4z = 49$

$\Rightarrow 2x - 3y - 4z = 49$

$\Rightarrow 2\left(\frac{\lambda}{2}\right) + 3\left(\frac{3\lambda}{2}\right) + 4\left(\frac{2\lambda}{3}\right) = 49$

$\Rightarrow \lambda + \frac{9\lambda}{2} + \frac{8\lambda}{3} = 49$

$\Rightarrow 6\lambda + 27\lambda + 16\lambda = 294$

$\Rightarrow \lambda = 6$

$x = 3, \ y = -9, \ z = -4$

$(3, -9, -4)$ IS THE ONLY CRIT POINT.

$$\boxed{f(3, -9, -4) = 147}$$

HOW DO WE KNOW THIS IS A MIN ?

NOTICE THAT $\left(\frac{49}{2}, 0, 0\right)$ SATISFIES THE CONSTRAINT AND

$f\left(\frac{49}{2}, 0, 0\right) = 1200.5$,

WHICH IS greater THAN 147.

Math 173 - Test 3b
April 28, 2011

Name key

Score ________

Show all work. Supply explanations when necessary.
YOU MUST WORK INDIVIDUALLY ON THIS EXAM.

1. (13 points) Find and classify all critical points of the following function.

$$f(x,y) = y^3 - 3x^2y - 3y^2 - 3x^2 + 1$$

$f_x(x,y) = -6xy - 6x = 0 \Rightarrow -6x(y+1) = 0 \Rightarrow x = 0$ OR $y = -1$

$f_y(x,y) = 3y^2 - 3x^2 - 6y$

$\underline{x=0}$

$3y^2 - 6y = 0$

$3y(y-2) = 0$

$y = 0, y = 2$

$\underline{y=-1}$

$3 - 3x^2 + 6 = 0$

$3x^2 = 9$

$x = \pm\sqrt{3}$

CRIT POINTS ARE $(0,0), (0,2), (\sqrt{3},-1), (-\sqrt{3},-1)$

$$d(x,y) = \begin{vmatrix} -6y-6 & -6x \\ -6x & 6y-6 \end{vmatrix} = -(6y+6)(6y-6) - 36x^2 = -36y^2 + 36 - 36x^2$$

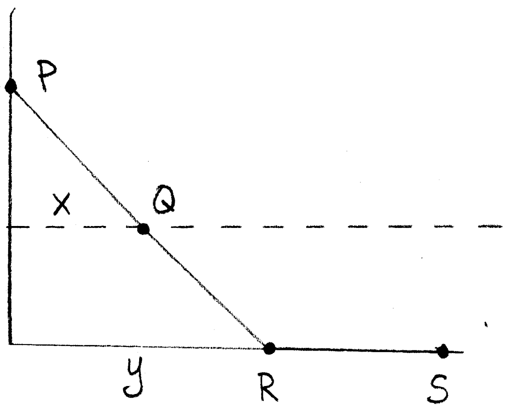
$d(0,0) = 36$ AND $f_{xx}(0,0) = -6 \Rightarrow$ $\boxed{f(0,0) = 1 \text{ IS A RELATIVE MAX}}$

$d(0,2) = -108 < 0$ AND $f(0,2) = -3 \Rightarrow$ $\boxed{(0,2,-3) \text{ IS A SADDLE POINT.}}$

$d(\sqrt{3},1) = -108$ AND $f(\sqrt{3},-1) = -3 \Rightarrow$ $\boxed{(\sqrt{3},-1,-3) \text{ IS A SADDLE POINT.}}$

$d(-\sqrt{3},1) = -108$ AND $f(-\sqrt{3},-1) = -3 \Rightarrow$ $\boxed{(-\sqrt{3},-1,-3) \text{ IS A SADDLE POINT.}}$

2. (10 points) Section 13.9, Page 967, Problem #19.



$$\text{Cost} = 3k\sqrt{x^2+4} + 2k\sqrt{(y-x)^2+1} + k(10-y) = C(x,y)$$

$$C_x(x,y) = \frac{3}{2}k(x^2+4)^{-1/2}(2x) + k\left[(y-x)^2+1\right]^{-1/2}(2)(y-x)(-1) = 0$$

$$C_y(x,y) = \underbrace{k\left[(y-x)^2+1\right]^{-1/2}(2)(y-x) + k(-1) = 0}$$

$$\frac{2(y-x)}{\sqrt{(y-x)^2+1}} = 1$$

Then $C_x(x,y) = 0 \Rightarrow \dfrac{3x}{\sqrt{x^2+4}} = 1 \Rightarrow 3x = \sqrt{x^2+4}$

$$9x^2 = x^2+4$$

$$8x^2 = 4 \Rightarrow x^2 = \frac{1}{2}$$

$$\Rightarrow x = \frac{1}{\sqrt{2}}$$

$$\approx 0.7071$$

$$2y - \frac{2}{\sqrt{2}} = \sqrt{\left(y-\frac{1}{\sqrt{2}}\right)^2+1}$$

CAS SOLUTION

$$y = \frac{2\sqrt{3}+3\sqrt{2}}{6} \approx 1.284457$$

See attached sheet for 2^{nd} partials test. Min cost is given by...

$$x = \frac{1}{\sqrt{2}} \approx 0.707 \text{ km}, \quad y = \frac{2\sqrt{3}+3\sqrt{2}}{6} \approx 1.284 \text{ km}$$

```
(%i1) C: 3*k*sqrt(x^2+4)+2*k*sqrt((y-x)^2+1)+k*(10-y);
```

(%o1) $2\,k\,\sqrt{(y-x)^2+1}+k\,(10-y)+3\,k\,\sqrt{x^2+4}$

```
(%i2) Cx: diff( C, x );
```

(%o2) $\dfrac{3\,k\,x}{\sqrt{x^2+4}}-\dfrac{2\,k\,(y-x)}{\sqrt{(y-x)^2+1}}$

```
(%i3) Cy: diff( C, y );
```

(%o3) $\dfrac{2\,k\,(y-x)}{\sqrt{(y-x)^2+1}}-k$

```
(%i4) Cxx: diff( Cx, x );
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(%o4) $\dfrac{2\,k}{\sqrt{(y-x)^2+1}}-\dfrac{2\,k\,(y-x)^2}{\left((y-x)^2+1\right)^{3/2}}+\dfrac{3\,k}{\sqrt{x^2+4}}-\dfrac{3\,k\,x^2}{\left(x^2+4\right)^{3/2}}$

```
(%i5) Cxy: diff( Cx, y );
```

(%o5) $\dfrac{2\,k\,(y-x)^2}{\left((y-x)^2+1\right)^{3/2}}-\dfrac{2\,k}{\sqrt{(y-x)^2+1}}$

```
(%i6) Cyy: diff( Cy, y );
```

(%o6) $\dfrac{2\,k}{\sqrt{(y-x)^2+1}}-\dfrac{2\,k\,(y-x)^2}{\left((y-x)^2+1\right)^{3/2}}$

(a,b) is the only critical point.

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(%i7) a: 1/sqrt(2); b: (2*sqrt(3)+3*sqrt(2))/6;
```

(%o7) $\dfrac{1}{\sqrt{2}}$

(%o8) $\dfrac{2\,\sqrt{3}+3\,\sqrt{2}}{6}$

D is positive at (a,b).

```
(%i11) float(at( Cxx, [x=a,y=b] )*at( Cyy, [x=a,y=b] ) - at( Cxy, [x=a,y=b]
```

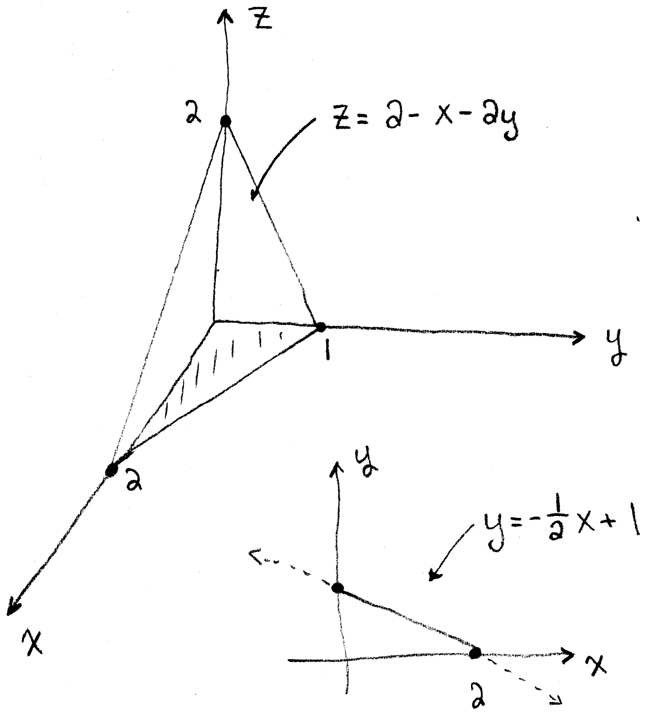
(%o11) $1.632993161855453\,k^2$

Cxx is positive at (a,b).

```
(%i12) float(at( Cxx, [x=a,y=b] ));
```

(%o12) $2.556116827786077\,k$

3. (17 points) A tetrahedron in the 1st octant is bounded by the coordinate planes and the plane $x + 2y + z = 2$. The density of the tetrahedron at the point (x, y, z) is given by $\rho(x, y, z) = 2 + 2x + y + z^2$. Sketch the tetrahedron. Then set up all integrals required to determine the center of mass. Use your calculator (or computer algebra system) to evaluate the integrals and find the center of mass.



$$M = \int_{x=0}^{x=2} \int_{y=0}^{y=-\frac{1}{2}x+1} \int_{z=0}^{z=2-x-2y} (2 + 2x + y + z^2)\, dz\, dy\, dx = \frac{73}{30}$$

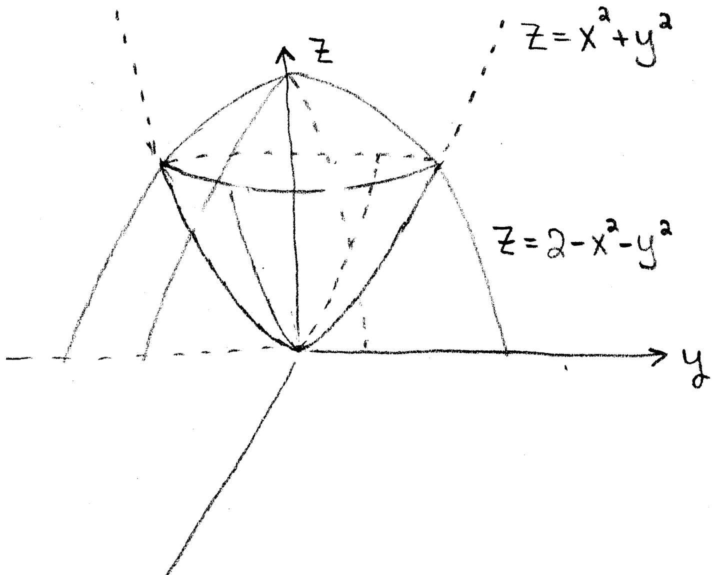
$$M_{yz} = \int_{x=0}^{x=2} \int_{y=0}^{y=-\frac{1}{2}x+1} \int_{z=0}^{z=2-x-2y} x(2 + 2x + y + z^2)\, dz\, dy\, dx = \frac{61}{45}$$

$$M_{xz} = \int_{x=0}^{x=2} \int_{y=0}^{y=-\frac{1}{2}x+1} \int_{z=0}^{z=2-x-2y} y(2 + 2x + y + z^2)\, dz\, dy\, dx = \frac{26}{45}$$

$$M_{xy} = \int_{x=0}^{x=2} \int_{y=0}^{y=-\frac{1}{2}x+1} \int_{z=0}^{z=2-x-2y} z(2 + 2x + y + z^2)\, dz\, dy\, dx = \frac{19}{15}$$

$$\text{Center of Mass} = \left(\frac{122}{219}, \frac{52}{219}, \frac{38}{73} \right)$$

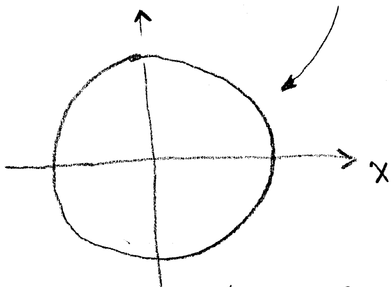
4. (10 points) Section 14.7, Page 1043, Problem #20. (Integrate by hand.)



x

$$x^2 + y^2 = 2 - x^2 - y^2$$

$$x^2 + y^2 = 1$$



$$\text{Volume} = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=r^2}^{z=2-r^2} r \, dz \, dr \, d\theta$$

$$= 2\pi \int_0^1 zr \Big|_{z=r^2}^{z=2-r^2} dr$$

$$= 2\pi \int_0^1 (2r - r^3 - r^3)\, dr = 2\pi \int_0^1 (2r - 2r^3)\, dr = 2\pi \left(r^2 - \tfrac{1}{2} r^4 \right) \Big|_0^1$$

$$= 2\pi \left(\tfrac{1}{2}\right) = \boxed{\pi}$$