# Math 173 - Quiz 4

February 24, 2011

Name key

Score ________

Show all work to receive full credit. Supply explanations when necessary.

1. (3 points) Find a point of intersection of the graphs of the following vector-valued functions.

$$\vec{r}(t) = (t-2)\hat{\imath} + t^2\hat{\jmath} + \frac{1}{2}t\hat{k}$$

$$\vec{u}(s) = \frac{1}{4}s\hat{\imath} + 2s\hat{\jmath} + \sqrt[3]{s}\hat{k}$$

Then find the angle between the tangent vectors at that point.

POINT OF INTERSECTION:

$$\underbrace{\begin{aligned} t-2 &= \tfrac{1}{4}s \\ t^2 &= 2s \end{aligned}}$$

$t^2 = 2(4t-8)$

$t^2 = 8t-16$

$t^2 - 8t + 16 = 0$

$(t-4)^2 = 0 \Rightarrow t = 4$

POINT IS $t=4$, $s=8$

WHICH CORRESPONDS TO $\boxed{(2, 16, 2)}$.

$\vec{r}\,'(t) = \hat{\imath} + 2t\hat{\jmath} + \frac{1}{2}\hat{k}$

$\vec{u}\,'(s) = \frac{1}{4}\hat{\imath} + 2\hat{\jmath} + \frac{1}{3}s^{-2/3}\hat{k}$

$\vec{r}\,'(4) = \hat{\imath} + 8\hat{\jmath} + \frac{1}{2}\hat{k}$

$\vec{S}\,'(8) = \frac{1}{4}\hat{\imath} + 2\hat{\jmath} + \frac{1}{12}\hat{k}$

$|\vec{r}\,'(4)| = \sqrt{261/4}$

$|\vec{S}\,'(8)| = \sqrt{293/72}$

$\vec{r}\,'(4)\cdot S'(8) = \frac{391}{24}$

$$\theta = \cos^{-1}\left(\frac{391/24}{\sqrt{\frac{261}{4}}\sqrt{\frac{293}{72}}}\right)$$

$$\boxed{= 0.0206\ldots \approx 1.18^\circ}$$

2. (3 points) Find the principal unit normal vector at $t=0$.

$$\vec{r}(t) = \sqrt{2}\,t\hat{\imath} + e^t\hat{\jmath} + e^{-t}\hat{k}$$

$\vec{r}\,'(t) = \sqrt{2}\,\hat{\imath} + e^t\hat{\jmath} - e^{-t}\hat{k}$

$|\vec{r}\,'(t)| = \sqrt{2 + e^{2t} + e^{-2t}}$

$$\hat{T}(t) = \frac{1}{\sqrt{2+e^{2t}+e^{-2t}}}\left(\sqrt{2}\,\hat{\imath} + e^t\hat{\jmath} - e^{-t}\hat{k}\right)$$

$$\hat{T}\,'(t) = -\frac{1}{2}\left(2+e^{2t}+e^{-2t}\right)^{-3/2}\left(2e^{2t}-2e^{-2t}\right)\left[\sqrt{2}\,\hat{\imath} + e^t\hat{\jmath} - e^{-t}\hat{k}\right]$$

$$+ \frac{1}{\sqrt{2+e^{2t}+e^{-2t}}}\left(e^t\hat{\jmath} + e^{-t}\hat{k}\right) \Rightarrow \hat{T}\,'(0) = \frac{\hat{\jmath}+\hat{k}}{2}$$

$$\hat{N}(0) = \frac{\hat{T}\,'(0)}{|\hat{T}\,'(0)|} = \frac{\frac{\hat{\jmath}+\hat{k}}{2}}{\frac{1}{\sqrt{2}}} = \boxed{\frac{\hat{\jmath}+\hat{k}}{\sqrt{2}}}$$

3. (2 points) Suppose $\vec{r}(t) = a\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath} + b\,\hat{k}$, where $a$ and $b$ are positive numbers. Find a formula for the arc length of the graph over the interval $[0, 2\pi]$.

$$\vec{r}\,'(t) = -a\sin t\,\hat{\imath} + a\cos t\,\hat{\jmath}$$

$$|\vec{r}\,'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = \sqrt{a^2} = a$$

$$\int_0^{2\pi} |\vec{r}\,'(t)|\,dt = \int_0^{2\pi} a\,dt = \boxed{2\pi a}$$

4. (2 points) Find the curvature of the graph of $y = 2x + \dfrac{4}{x}$ at the point where $x = 1$.

$$f(x) = 2x + \frac{4}{x}$$

$$f'(x) = 2 - \frac{4}{x^2} \qquad f'(1) = -2$$

$$f''(x) = \frac{8}{x^3} \qquad f''(1) = 8$$

$$K = \frac{|f''(x)|}{[1 + f'(x)^2]^{3/2}} = \frac{8}{5^{3/2}} = \boxed{0.7155\ldots}$$