Math 172 - Test 1
September 20, 2017

Name Key

Score ________

Show all work to receive full credit. Supply explanations where necessary.

1. (8 points) Find an equation of the line tangent to the graph of $y = \dfrac{e^{2x} + e^{-x^3}}{2}$ at the point where $x = 1$. Round your numbers to two decimal places.

$$\frac{dy}{dx} = \frac{2e^{2x} - 3x^2 e^{-x^3}}{2}, \quad m = \left.\frac{dy}{dx}\right|_{x=1} = \frac{2e^2 - 3e^{-1}}{2} \approx 6.84$$

$$y\Big|_{x=1} = \frac{e^2 + e^{-1}}{2} \approx 3.88$$

TAN LINE: $y - 3.88 = 6.84(x-1)$

2. (6 points) Solve each equation. Round your answers to three decimal places.

(a) $2^{5-x} = 750$

$$(5-x)\ln 2 = \ln 750$$

$$5 - x = \frac{\ln 750}{\ln 2} \Rightarrow x = 5 - \frac{\ln 750}{\ln 2} \approx -4.551$$

(b) $\log_3 x^2 = 4.5$

$$x^2 = 3^{4.5} \Rightarrow |x| = \sqrt{3^{4.5}} \approx 11.845$$

$$\Rightarrow x \approx \pm 11.845$$

3. (10 points) For $x > 2/3$, let $y = \dfrac{x^2\sqrt{3x-2}}{(x+1)^2}$. Find dy/dx.

$$\ln y = 2\ln x + \frac{1}{2}\ln(3x-2) - 2\ln(x+1)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{1}{2}\,\frac{3}{3x-2} - \frac{2}{x+1}$$

$$\frac{dy}{dx} = \frac{x^2\sqrt{3x-2}}{(x+1)^2}\left[\frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x+1}\right]$$

4. (14 points) Evaluate the indefinite integral: $\displaystyle\int \frac{x}{x^2-4x+13}\,dx$

$u = x^2-4x+13$
$du = (2x-4)\,dx$
$\frac{1}{2}du = (x-2)\,dx$

$$\int \frac{x-2}{x^2-4x+13}\,dx + \int \frac{2}{x^2-4x+13}\,dx$$

$(x^2-4x+4+9)$

$$= \frac{1}{2}\int \frac{1}{u}\,du + \int \frac{2}{(x-2)^2+9}\,dx$$

$$= \frac{1}{2}\ln|x^2-4x+13| + \frac{2}{3}\tan^{-1}\frac{x-2}{3} + C$$

5. (8 points) Consider the function $f(x) = \sqrt{4 - x^2}$, $0 \le x \le 2$.

(a) Algebraically determine $f^{-1}(x)$.

$y = \sqrt{4 - x^2}$, $0 \le x \le 2$

$y^2 = 4 - x^2$

$x^2 = 4 - y^2$

$|x| = \sqrt{4 - y^2}$

$x = \pm\sqrt{4 - y^2}$

RANGE OF INVERSE = DOMAIN OF ORIGINAL

$\Rightarrow$ TAKE + SQUARE ROOT

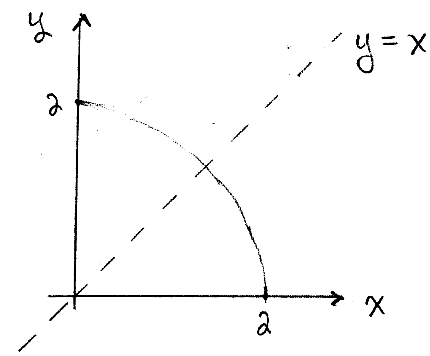
$x = \sqrt{4 - y^2} \Rightarrow f^{-1}(x) = \sqrt{4 - x^2}$, $0 \le x \le 2$

INTERCHANGE X,Y

$x^2 + y^2 = 4$

IN 1^{ST} QUAD

(b) Sketch the graph of $f(x)$ along with the graph of the line $y = x$.



(c) Based on your graph, explain the relationship that you notice between f and f^{-1}.

$f^{-1} = f$

THE GRAPH OF f IS ITS OWN REFLECTION ABOUT $y = x$, I.E. THE GRAPH IS SYMMETRIC ABOUT $y = x$.

6. (8 points) Evaluate the integral: $\int x\, 5^{-x^2}\, dx$.

$u = -x^2$

$du = -2x\, dx$

$-\frac{1}{2}\, du = x\, dx$

$-\frac{1}{2}\int 5^u\, du = -\frac{1}{2}\int e^{u \ln 5}\, du$

$= -\frac{1}{2\ln 5} e^{u \ln 5} + C$

$= -\frac{1}{2\ln 5} 5^{-x^2} + C$

7. (8 points) Find the exact value of each expression. Show all work and/or explain your reasoning. (You may refer to your trig unit circle.)

(a) $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ BECAUSE $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

AND $-\frac{\pi}{6}$ IS IN $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(b) $\tan^{-1}\left(\tan\frac{7\pi}{6}\right) = \tan^{-1}\left(\tan\frac{\pi}{6}\right)$ BECAUSE TANGENT IS PERIODIC WITH PERIOD π

$= \frac{\pi}{6}$

(c) $\csc^{-1}(\sqrt{2})$

$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$ BECAUSE $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

AND $\frac{\pi}{4}$ IS IN $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(d) $\cos^{-1}(0)$

$= \frac{\pi}{2}$ BECAUSE $\cos\frac{\pi}{2} = 0$ AND

$\frac{\pi}{2}$ IS IN $[0, \pi]$

8. (8 points) Let $h(x) = 7 - x - x^3$.

(a) Explain how we can be certain that h has an inverse

$h'(x) = -1 - 3x^2$

$h'(x) < 0$ FOR ALL X $\Rightarrow$ h IS DECREASING

DECREASING FUNCTIONS ARE 1-1.

1-1 FUNCTIONS HAVE INVERSES.

(b) Compute $(h^{-1})'(-3)$.

$= \frac{1}{h'(h^{-1}(-3))} = \frac{1}{h'(2)} = \frac{1}{-1-3(4)} = -\frac{1}{13}$

$h^{-1}(-3) = x \Leftrightarrow h(x) = -3$

$7 - x - x^3 = -3$

$\Leftrightarrow x = 2$

9. (8 points) Find dy/dx.

(a) $y = 3^{-4x}$

$$\ln y = -4x \ln 3$$

$$\frac{1}{y}\frac{dy}{dx} = -4\ln 3$$

$$\boxed{\frac{dy}{dx} = (-4\ln 3)\, 3^{-4x}}$$

(b) $y = x^{\sin x}$

$$\ln y = (\sin x)\ln x$$

$$\frac{1}{y}\frac{dy}{dx} = (\cos x)\ln x + (\sin x)\left(\frac{1}{x}\right)$$

$$\boxed{\frac{dy}{dx} = x^{\sin x}\left(\ln x \cos x + \frac{\sin x}{x}\right)}$$

10. (8 points) Evaluate the definite integral: $\displaystyle\int_1^3 \frac{e^{3/x}}{x^2}\,dx.$

$$u = \frac{3}{x}$$

$$du = -\frac{3}{x^2}\,dx$$

$$-\frac{1}{3}du = \frac{1}{x^2}\,dx$$

$$-\frac{1}{3}\int_{u=3}^{u=1} e^u\,du = -\frac{1}{3}e^u\Big|_3^1 = -\frac{1}{3}\left[e - e^3\right] \approx 5.789$$

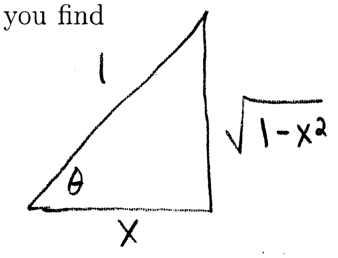
11. (6 points) Determine the derivative of $\sin(\cos^{-1} x)$. (3 extra credit points if you find the derivative two different ways.)

$$\frac{d}{dx} \sin\left(\cos^{-1} x\right)$$

$$= \cos\left(\cos^{-1} x\right)\left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$$= \frac{-x}{\sqrt{1-x^2}}$$

$$\sin(\underbrace{\cos^{-1} x}_{\theta}) = \sqrt{1-x^2}$$



$$\frac{d}{dx} \sin\left(\cos^{-1} x\right)$$

$$= \frac{d}{dx}\sqrt{1-x^2}$$

$$= \frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x)$$

$$= \frac{-x}{\sqrt{1-x^2}}$$

12. (8 points) Find the area of the 1st quadrant region under the graph of $y = \dfrac{x^2+4}{x}$ over the interval from $x = 1$ to $x = 4$.

$$\int_1^4 \frac{x^2+4}{x}\,dx = \int_1^4 \left(x + \frac{4}{x}\right) dx$$

$$= \frac{1}{2}x^2 + 4\ln x \Big|_1^4$$

$$= 8 + 4\ln 4 - \frac{1}{2}$$

$$\approx 13.045$$