Math 157 - Final Exam

December 10, 2014

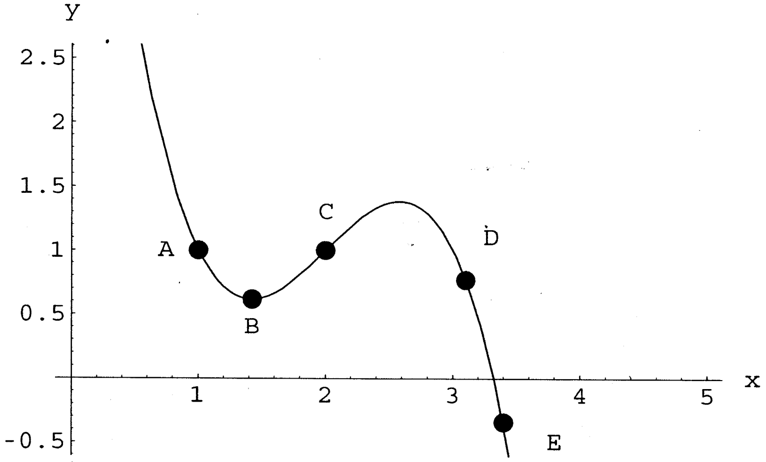
Name key

Score

Show all work to receive full credit. Supply explanations where necessary. Unless otherwise indicated, all limits, derivatives, and integrals should be computed by hand with all work shown.

---

1. (6 points) The graph of f is shown below. For each part of this problem, find a labeled point that satisfies the given condition.



(a) $f''(x) = 0$

C

(b) $f'(x) = 0$

B

(c) $f''(x) < 0$

D or E

(d) $f(x) < 0$

E

(e) $f'(x) > 0$

C

(f) $f''(x) > 0$

A or B

2. (6 points) Use a table of values to estimate the following limit. Your table must show function values at four or more points.

$$\lim_{x \to 4} \frac{x^2 - 4x}{2 - \sqrt{x}}$$

X	$\frac{x^2-4x}{2-\sqrt{x}}$
3.9	-15.5019
3.99	-15.9500
3.999	-15.9950
4.1	-16.5019
4.01	-16.0500
4.001	-16.0050

LOOKS LIKE LIMIT IS ABOUT $\boxed{-16}$

3. (8 points) Use algebra to evaluate the limit.

$$\lim_{x \to 0} \frac{(2+x)^2 - 4}{x} = \lim_{x \to 0} \frac{4+4x+x^2-4}{x}$$

$$= \lim_{x \to 0} \frac{4x+x^2}{x} = \lim_{x \to 0} (4+x)$$

$$= \boxed{4}$$

4. (8 points) Let $g(x) = 3^x$. Use at least three small intervals to estimate $g'(1)$.

$$[0.9, 1.1] \longrightarrow g'(1) \approx \frac{g(1.1) - g(0.9)}{1.1 - 0.9} \approx 3.302$$

$$[0.99, 1.01] \longrightarrow g'(1) \approx \frac{g(1.01) - g(0.99)}{1.01 - 0.99} \approx 3.296$$

$$[0.999, 1.001] \longrightarrow g'(1) \approx \frac{g(1.001) - g(0.999)}{1.001 - 0.999} \approx 3.296$$

IT LOOKS LIKE $\boxed{g'(1) \approx 3.296}$

5. (4 points) Evaluate the limit: $\lim_{t \to 0} \left[(t+2)e^{3t} + \ln(t+2)\right]$

$$= (0+2)e^{3(0)} + \ln(0+2)$$

$$= \boxed{2 + \ln 2 \approx 2.693}$$

6. (8 points) A glass of water is placed in a hot, dry room where the water begins to quickly evaporate. The height, in centimeters, of the water in the glass after t hours is given by

$$h(t) = 20 - 0.094t^2.$$

Compute $h(6)$ and $h'(6)$. Using units, explain what each of these values represents.

$h(6) = \boxed{16.616 \text{ cm} \leftarrow \text{HEIGHT OF WATER AFTER 6 HOURS}}$

$h'(t) = 0 - 0.094(2)t$

$h'(6) = \boxed{-1.128 \ \text{cm}/\text{HR} \leftarrow \text{AFTER 6 HOURS, THE HEIGHT IS DECREASING AT } 1.128 \ \text{cm}/\text{HR}}$

7. (10 points) Let $f(x) = (x^2 - x + 1)^2$. Find an equation of the line tangent to the graph of f at the point where $x = 2$.

SLOPE:

$f'(x) = 2(x^2 - x + 1)(2x - 1)$

$f'(2) = 2(3)(3) = 18$

POINT:

$x = 2 \Rightarrow y = f(2) = 3^2 = 9$

TANGENT LINE:

$y - 9 = 18(x - 2)$

OR

$y = 18x - 27$

8. (8 points) Find the second derivative of $y = 3x + 5\ln x$

$$\frac{dy}{dx} = 3 + \frac{5}{x} = 3 + 5x^{-1}$$

$$\boxed{\frac{d^2y}{dx^2} = -5x^{-2} = -\frac{5}{x^2}}$$

9. (10 points) Find the instantaneous rate of change of $y = x^2e^{5x}$ at the point where $x = -1$.

$$\frac{dy}{dx} = 2xe^{5x} + 5x^2e^{5x}$$

$$\left.\frac{dy}{dx}\right|_{x=-1} = -2e^{-5} + 5e^{-5} = \boxed{3e^{-5} \approx 0.02021}$$

10. (8 points) The profit P (in dollars) for producing x units of a product is given by $P = -2x^2 + 72x - 145$. Find the production level at which the marginal profit is zero.

$$P' = -4x + 72$$

$$P' = 0 \Rightarrow x = \frac{-72}{-4}$$

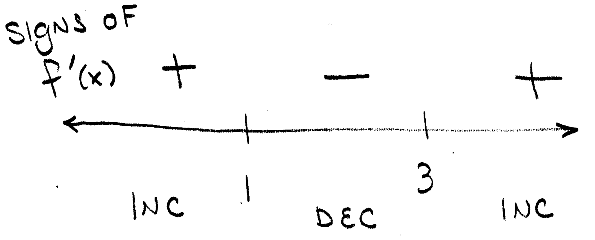
$$\boxed{x = 18}$$

11. (20 points) Consider the function $f(x) = 2x^3 - 12x^2 + 18x$.

(a) Find all critical numbers of f.

$$f'(x) = 6x^2 - 24x + 18$$
$$= 6(x^2 - 4x + 3)$$
$$= 6(x-3)(x-1) = 0$$
$$\Rightarrow \boxed{x = 3 \text{ or } x = 1}$$

(b) Find open intervals on which f is increasing/decreasing.



f IS INCREASING ON
$(-\infty,-1) \cup (3,\infty)$

f IS DECREASING ON $(1,3)$

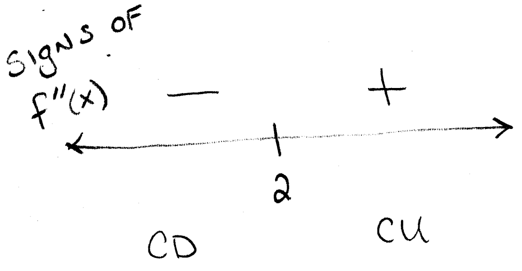
(c) Find the local (relative) extreme values of f.

$f(1) = 8$ IS A LOCAL MAX

$f(3) = 0$ IS A LOCAL MIN

(d) Find open intervals on which the graph of f is concave up/down.

$$f''(x) = 12x - 24 = 0$$
$$\Rightarrow x = 2$$



GRAPH IS CONCAVE DOWN
ON $(-\infty, 2)$
AND CONCAVE UP ON $(2,\infty)$

12. (10 points) The revenue from selling q items is $R(q) = 500q - q^2$, and the total cost is $C(q) = 150 + 10q$. Find the quantity that maximizes profit.

$$P(q) = 500q - q^2 - 150 - 10q$$

$$= -q^2 + 490q - 150$$

$$P'(q) = -2q + 490 = 0$$

$$\Rightarrow q = 245$$

$$P''(q) = -2 \Rightarrow \text{GRAPH IS CD}$$

$$\Rightarrow \boxed{q = 245 \text{ gives a max}}$$

13. (6 points) When the production level is 7000 units, marginal revenue is \$8.30 per unit and marginal cost is \$8.05 per unit. Do you expect maximum profit to occur at a production level above or below 7000 units? Explain your reasoning.

$$P'(7000) = R'(7000) - C'(7000)$$

$$= 8.30 - 8.05 = 0.25$$

$P'(7000) = 0.25 > 0 \Rightarrow$ PROFIT IS INCREASING $\Longrightarrow$ MAX PROFIT AT LEVEL ABOVE 7000

14. (8 points) Use a right sum with 5 subintervals (rectangles) of equal width to estimate $\int_0^1 \ln(x+1)\,dx$.

$$\Delta x = \frac{1-0}{5} = 0.2$$

PARTITION IS $0 < 0.2 < 0.4 < 0.6 < 0.8 < 1$

$$\text{RIGHT SUM} = 0.2(\ln 1.2 + \ln 1.4 + \ln 1.6 + \ln 1.8 + \ln 2)$$

$$\approx \boxed{0.4539}$$

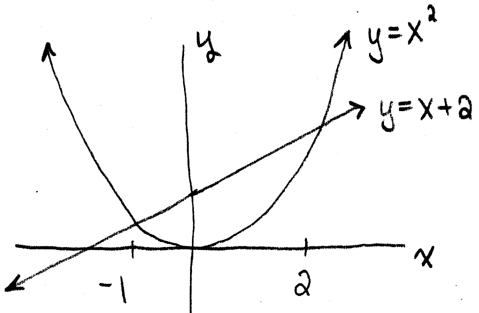
15. (12 points) Evaluate the definite integral: $\int_1^2 \left(6x^2 - \frac{5}{x} + 2e^x\right)\, dx$

$$= 2x^3 - 5\ln x + 2e^x \Big|_1^2$$

$$= \left(16 - 5\ln 2 + 2e^2\right) - \left(2 + 2e\right)$$

$$\approx \boxed{19.876}$$

16. (10 points) Set up the definite integral that gives the area of the bounded region between the graphs of $y = x^2$ and $y = x + 2$. Find the area by evaluating the integral. (You may use your calculator to evaluate your definite integral.)



$$\text{Area} = \int_{-1}^{2} (x + 2 - x^2)\, dx$$

$$\approx 4.5$$

$$x^2 = x+2$$

$$x^2 - x - 2 = (x-2)(x+1) = 0$$

$$x = 2,\ x = -1$$

17. (8 points) Evaluate the indefinite integral: $\int 4x^3 \left(x^4 + 8\right)^5 dx$.

$$u = x^4 + 8$$

$$du = 4x^3\, dx$$

$$\int u^5\, du = \frac{1}{6}u^6 + C$$

$$= \boxed{\frac{1}{6}\left(x^4+8\right)^6 + C}$$

18. (Extra credit) You may do any two of these problems for up to 12 extra credit points.

(a) Evaluate the definite integral: $\int_0^1 \frac{x}{1+x^2}\,dx$

$u = 1+x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln u + C$

$= \frac{1}{2}\ln(1+x^2) + C$

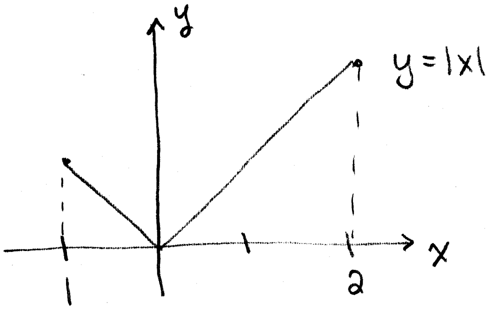
$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\ln(1+x^2)\Big|_0^1 = \boxed{\frac{1}{2}\ln 2}$$

(b) Find the second derivative of $y = \frac{5x - x^3}{7+x}$.

$$\frac{dy}{dx} = \frac{(7+x)(5-3x^2) - (5x-x^3)(1)}{(7+x)^2} = \frac{35 - 21x^2 - 2x^3}{(7+x)^2}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{(7+x)^2(-42x-6x^2) - (35-21x^2-2x^3)(2)(7+x)}{(7+x)^4}}$$

(c) Sketch the graph of $y = |x|$. Use area to compute the exact value of $\int_{-1}^{2} |x|\,dx$



$$\int_{-1}^{2} |x|\,dx$$

$= \frac{1}{2}(1)(1) + \frac{1}{2}(2)(2)$

$= \frac{1}{2} + 2 = \boxed{2.5}$