Math 157 - Test 2

October 15, 2014

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (8 points) Find an equation of the line tangent to the graph of

$$f(x) = x^3 - 5x^2 + 4x + 6$$

at the point where $x = 1$.

Slope: $f'(x) = 3x^2 - 10x + 4$

$m = f'(1) = 3 - 10 + 4 = -3$

Point: $x = 1$, $y = f(1) = 1 - 5 + 4 + 6$
$= 6$

$y - 6 = -3(x-1)$

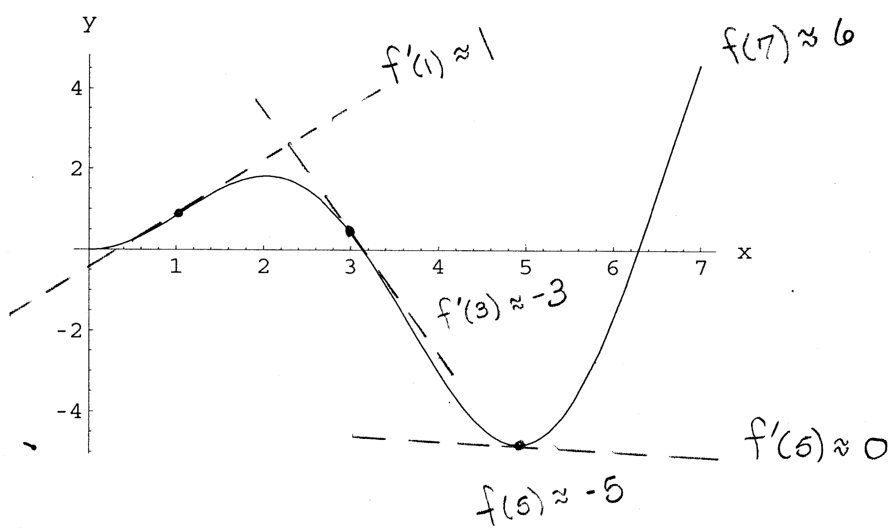
or

$y = -3x + 9$

2. (5 points) The graph of the function f is shown below. Referring to this graph, arrange the following quantities in ascending order.

$$f'(1), \quad f'(5), \quad f(5), \quad f(7), \quad f'(3)$$

Ascending order: $f(5), f'(3), f'(5), f'(1), f(7)$



3. (20 points) Determine the derivative of each function.

(a) $y = 8t^3 - 4t^2 + 12t - 3$

$$\boxed{\frac{dy}{dt} = 24t^2 - 8t + 12}$$

(b) $f(x) = \sqrt{\frac{1}{x^3}} = x^{-3/2}$

$$\boxed{f'(x) = -\frac{3}{2}x^{-5/2}}$$

(c) $P = \ln(t^2 + 1)$

$$\frac{dP}{dt} = \frac{1}{t^2+1}(2t) = \boxed{\frac{2t}{t^2+1}}$$

(d) $w = (5r^2 - 8)^4$

$$\frac{dw}{dr} = 4(5r^2-8)^3(10r) = \boxed{40r(5r^2-8)^3}$$

(e) $g(x) = (x^3 - 7x^2 + 1)e^x$

$$\boxed{g'(x) = (x^3-7x^2+1)e^x + (3x^2-14x)e^x}$$

4. (7 points) A glass of water is placed in a hot, dry room where the water begins to quickly evaporate. The height, in centimeters, of the water in the glass after t hours is given by

$$h(t) = 20 - 0.094t^2.$$

Compute $h(6)$ and $h'(6)$. Using units, explain what each of these values represents.

$\boxed{h(6) = 16.616 \text{ cm}}$ = HEIGHT OF WATER IN CM AFTER 6 HRS

$h'(t) = -0.094(2t) = -0.188t$

$\boxed{h'(6) = -1.128 \text{ cm/HR}}$ → AT THE 6 HR MARK, THE HEIGHT OF WATER IS DECREASING AT 1.128 cm/HR

5. (6 points) Given that $f(1) = 4$ and $f'(1) = 5$, find $g'(1)$ if $g(x) = \sqrt{f(x)}$.

$g(x) = [f(x)]^{1/2}$ $\quad g'(x) = \underbrace{\frac{1}{2}[f(x)]^{-1/2}(f'(x))}_{\text{CHAIN RULE!}}$

$g'(1) = \frac{1}{2}[f(1)]^{-1/2} f'(1)$

$= \frac{5}{4}$

6. (7 points) Find the critical numbers of $f(x) = (x^2 - 4)^3$.

DEFINED FOR ALL REAL #'S

$f'(x) = 3(x^2-4)^2(2x) = 0$

$= 3(x-2)^2(x+2)^2(2x) = 0$

SETTING FACTORS EQUAL TO ZERO GIVES

$x = 2, \; x = -2, \; x = 0$

$f'(x)$ DNE NOWHERE

ONLY CRIT #'S ARE $x = 2, \; x = -2, \; x = 0$

7. (7 points) Since the beginning of 1990, the population of Aurora, IL has been approximately given by

$$P(t) = 101(1.03)^t,$$

where t represents years since 1990 and P is in thousands of people. At what rate will the population be changing in 2025? Include units with your answer.

$\frac{d}{dt} a^t = (\ln a) a^t$

$$P'(t) = 101(1.03)^t \ln(1.03)$$

2025 is $t = 35$

$$P'(35) \approx 8.4$$

AT THE BEGINNING OF 2025, THE POPULATION WILL BE INCREASING AT 8400 PEOPLE PER YEAR

8. (12 points) Find the second derivative of each function.

(a) $h(x) = x^4 + 5x^2 + e^{-3x}$

$$h'(x) = 4x^3 + 10x - 3e^{-3x}$$

$$\boxed{h''(x) = 12x^2 + 10 + 9e^{-3x}}$$

(b) $r(x) = 2x + 2\ln x$

$$r'(x) = 2 + \frac{2}{x} = 2 + 2x^{-1}$$

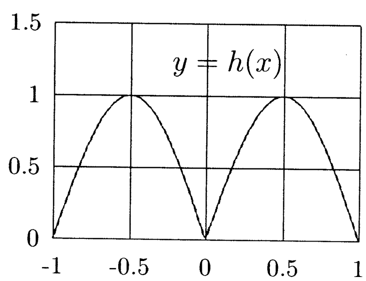
$$\boxed{r''(x) = -2x^{-2}}$$

9. (7 points) Find the instantaneous rate of change of $y = x^2 e^{5x}$ at the point where $x = -1$.

$$\frac{dy}{dx} = 2xe^{5x} + x^2(5e^{5x})$$

$$\left.\frac{dy}{dx}\right|_{x=-1} = -2e^{-5} + 5e^{-5} = \boxed{3e^{-5} \approx 0.020214}$$

10. (7 points) Consider the function $y = h(x)$ whose graph is shown below. The domain of h is the closed interval $[-1, 1]$. Find the critical numbers of h and explain why they are critical numbers.



DOMAIN INTERIOR PTS AT WHICH $h'(x) = 0$

ARE $\underbrace{X = -0.5 \text{ AND } X = 0.5}_{\text{HOR. TANGENT LINES}}$

DOMAIN INTERIOR PT AT WHICH $h'(x)$ DNE

IS $\underbrace{X = 0}_{\text{SHARP POINT}}$

CRIT #'S ARE $X = -0.5$, $X = 0.5$, $X = 0$

11. (8 points) The table below gives the values of the functions f and g and their derivatives at selected values of x.

x	-2	-1	2
$f(x)$	1	3	-2
$f'(x)$	2	-1	-1
$g(x)$	2	0	-2
$g'(x)$	-3	-2	1

(a) If $h(x) = f(x) \cdot g(x)$, use the product rule to compute $h'(-1)$.

$$h'(x) = f'(x)g(x) + f(x)g'(x)$$

$$h'(-1) = f'(-1)g(-1) + f(-1)g'(-1)$$

$$= (-1)(0) + (3)(-2) = \boxed{-6}$$

(b) If $h(x) = \dfrac{f(x)}{g(x)}$, use the quotient rule to compute $h'(2)$.

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$h'(2) = \frac{(-2)(-1) - (-2)(1)}{(-2)^2} = \frac{4}{4} = \boxed{1}$$

12. (6 points) For a function $f(x)$, we know that $f(10) = 8$ and $f'(10) = 3$. Use this information to find a reasonable estimate for $f(9)$.

$$f'(10) \approx \frac{f(10) - f(9)}{10-9} \approx 3 \Rightarrow 8 - f(9) \approx 3$$

$$\boxed{f(9) \approx 5}$$

13. (5 points extra credit) Find the critical numbers of $g(x) = (x-2)^2(2x-9)$.

$$g'(x) = 2(x-2)(1)(2x-9) + (x-2)^2(2)$$

$$= 2(x-2)\left[2x-9 + x-2\right]$$

$$= 2(x-2)(3x-11) = 0$$

$$\Rightarrow \boxed{x = 2 \text{ or } x = \frac{11}{3}}$$

14. (5 points extra credit) Find the second derivative of $w = \dfrac{3y - y^2}{5+y}$.

$$\frac{dw}{dy} = \frac{(5+y)(3-2y) - (3y-y^2)(1)}{(5+y)^2}$$

$$= \frac{15 - 10y - y^2}{(5+y)^2}$$

$$\boxed{\frac{d^2w}{dy^2} = \frac{(5+y)^2(-10-2y) - (15-10y-y^2)(2)(5+y)}{(5+y)^4}}$$