Problem 1

Find the radius and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$. (Be sure to also check for convergence at the interval endpoints).

Ratio test...

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2}\cdot\frac{n^2}{x^n}\right| = \lim_{n\to\infty}\frac{n^2|x|}{(n+1)^2} = |x|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = |x|\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^2 = |x|\cdot 1$$

By the ratio test, the series converges absolutely when this limit is less than 1. So, right now, we have convergence when $|x| < 1$. The radius of convergence is 1, and the current interval is $-1 < x < 1$.

Check the interval endpoints...

$$x = -1 \qquad \longrightarrow \qquad \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

The series of absolute values is the convergent $p = 2$ series. So this series converges absolutely.

$$x = 1 \qquad \longrightarrow \qquad \sum_{n=1}^{\infty}\frac{1}{n^2}$$

This series is the convergent $p = 2$ series. So this series converges.

Final answers...

$$\begin{gathered}\text{Radius of convergence: } R = 1\\ \text{Interval of convergence: } -1 \le x \le 1\end{gathered}$$

Problem 2

Find the interval of convergence of the power series $\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n$. (Be sure to also check for convergence at the interval endpoints).

This is a geometric series with ratio $r = x/4$. It converges IF AND ONLY IF

$$\left|\frac{x}{4}\right| < 1.$$

It follows that the series converges if and only if

$$|x| < 4 \qquad \text{or} \qquad -4 < x < 4.$$

Because we used the geometric series test, there is no need to check the endpoints. (The geometric series diverges when $|r| = 1$.)

Final answer...

$$\text{Interval of convergence: } -4 < x < 4$$

Problem 3

Consider the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Part (a)

Show that the radius of convergence is ∞.

Ratio test...

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = |x| \lim_{n\to\infty} \frac{1}{n+1} = |x| \cdot 0 = 0.$$

The limit is always less than 1, so by the ratio test, the series converges absolutely for all x.

Part (b)

Since the power series converges for all x, it converges to some function $f(x)$ that is defined for all real numbers. Use term-by-term differentiation to find a power series for $f'(x)$.

$$f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$$

Part (c)

Use term-by-term integration to find a power series for $\int f(x)\, dx$.

$$\int f(x)\, dx = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)n!} = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}$$

Now, if you replace the constant of integration C with the new constant of integration $C+1$, you will find

$$\int f(x)\, dx = C + 1 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} = C + \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

so that

$$\int f(x)\, dx = f(x) + C.$$

Part (d)

Based on your results, can you determine a more familiar expression for $f(x)$?

Based and parts (b) and (c), it sure looks like $f(x) = e^x$.