# Math 131 - Final Exam
December 11–12, 2024

Name key

Score ______

Show all work to receive full credit. Supply explanations where necessary.

1. (10 points) Use algebraic techniques (not a graph, table, or L'Hôpital's rule) to determine each limit.

   (a) $\lim_{x\to -1} \frac{x^3 - x}{x^2+3x+2}$

   $0/0$

   $= \lim_{x\to -1} \frac{x\cancel{(x+1)}(x-1)}{\cancel{(x+1)}(x+2)} = \lim_{x\to -1} \frac{x(x-1)}{x+2}$

   $= \boxed{2}$

   (b) $\lim_{x\to\infty} \frac{\sqrt{4x^2+9}}{x+3}$

   $\infty/\infty$

   $\cdot \frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{4+\frac{9}{x^2}}}{1+\frac{3}{x}} = \frac{\sqrt{4}}{1} = \boxed{2}$

2. (10 points) Each function below has a single discontinuity. For each function, find the point of discontinuity and state the kind of discontinuity (removable, jump, or infinite). For full credit, you must show some work.

   (a) $f(x) = \frac{\sin(x-2)}{x-2}$

   $\boxed{\text{DISCONT AT } X = 2.}$

   $\lim_{x\to 2} \frac{\sin(x-2)}{x-2} = 1$

   $\Rightarrow \boxed{\text{REMOVABLE}}$

   (b) $g(x) = \frac{x^2+1}{x+1}$

   $\boxed{\text{DISCONT AT } X = -1.}$

   $2/0$

   $\lim_{x\to -1} \frac{x^2+1}{x+1}$ DNE UNBOUNDED GROWTH

   $\boxed{\text{INFINITE}}$

   (c) $h(x) = \frac{x^2+2x}{|x|}$

   $\boxed{\text{DISCONT AT } X = 0.}$

   $\lim_{x\to 0^+} \frac{x^2+2x}{|x|} = \lim_{x\to 0} \frac{x^2+2x}{x} = 2$

   $\lim_{x\to 0^-} \frac{x^2+2x}{|x|} = \lim_{x\to 0} \frac{x^2+2x}{-x} = -2$

   $\boxed{\text{Jump}}$

3. (10 points) A potato is launched vertically upward with an initial velocity of 48 ft/s from a potato gun at the top of an 160-foot-tall building. After $t$ seconds, the potato's height (in feet) measured from the ground is given by $s(t) = -16t^2 + 48t + 160$.

(a) Determine the maximum height of the potato.

$$s'(t) = -32t + 48 = 0 \Rightarrow t = \frac{48}{32} = \frac{3}{2}$$

$$s\left(\frac{3}{2}\right) = -16\left(\frac{9}{4}\right) + 48\left(\frac{3}{2}\right) + 160$$

$$= -36 + 72 + 160 = \boxed{196 \text{ FT}}$$

(b) Determine the velocity of the potato when it hits the ground.

$$s(t) = 0 \Rightarrow -16(t^2 - 3t - 10) = 0$$

$$-16(t-5)(t+2) = 0 \Rightarrow t = 5$$

$$s'(5) = -32(5) + 48 = \boxed{-112 \text{ FT/s}}$$

4. (10 points) Use calculus techniques to find the absolute maximum and minimum values of $f(x) = 5x^2 - 6x^{5/3}$ on the interval $[0, 2]$.

$$f'(x) = 10x - 10x^{2/3}$$

$$x - x^{2/3} = 0$$

$$\Downarrow$$

$$x = x^{2/3}$$

$$\Downarrow$$

$$x = 0, 1$$

CRIT # IS $x = 1$

ENDPTS : $x = 0, x = 2$

| $x$ | $f(x)$ |
|---|---|
| 1 | $-1$ |
| 0 | 0 |
| 2 | $20 - 6\sqrt[3]{32} \approx 0.951$ |

$$\boxed{\begin{array}{ll} f(2) \approx 0.951 & \text{ABS MAX} \\ f(1) = -1 & \text{ABS MIN} \end{array}}$$

5. (10 points) The function $f$ and its first two derivatives are shown below.

$$f(x) = \frac{x}{x^2+9}$$

$$f'(x) = -\frac{x^2-9}{(x^2+9)^2} \qquad f''(x) = \frac{2x(x^2-27)}{(x^2+9)^3}$$

THESE DNE NOWHERE !

(a) Find all relative extreme values of $f$.

$f'(x) = 0 \Rightarrow x^2 - 9 = 0$

$\Rightarrow (x+3)(x-3) = 0$

$x = -3, \; x = 3$

SIGNS OF $f'$: $-$ on $(-\infty, -3)$, $+$ on $(-3, 3)$, $-$ on $(3, \infty)$

$f$ IS: DEC, INC, DEC

$$\boxed{\begin{array}{lll} f(-3) = \frac{-3}{18} = -\frac{1}{6} & \text{REL MIN} \\ f(3) = \frac{3}{18} = \frac{1}{6} & \text{REL MAX} \end{array}}$$

(b) Find open intervals on which the graph of $f$ is concave up/down.

$f''(x) = 0 \Rightarrow 2x(x^2 - 27) = 0$

$x = 0, \; x = \pm\sqrt{27}$

SIGNS OF $f''$: $-$, $+$, $-$, $+$ (at $-\sqrt{27}$, $0$, $\sqrt{27}$)

GRAPH IS: CD, CU, CD, CU

GRAPH IS CONCAVE UP ON $(-\sqrt{27}, 0) \cup (\sqrt{27}, \infty)$

GRAPH IS CONCAVE DOWN ON $(-\infty, -\sqrt{27}) \cup (0, \sqrt{27})$

6. (10 points) Use any analytical method (not a table or graph) to determine each limit.

(a) $\lim_{x\to 0^+} \sqrt{x}\ln x$ $\quad 0\cdot\infty$

$$= \lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} \overset{\infty/\infty}{=} \underbrace{\lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}}}_{\text{L'Hôpital's Rule}} = \lim_{x\to 0^+} -2x^{1/2} = \boxed{0}$$

(b) $\lim_{x\to 0}\left(\frac{e^x-(1+x)}{x^2}\right)$

$$\overset{0/0}{=} \underbrace{\lim_{x\to 0} \frac{e^x-1}{2x}}_{\text{L'Hôpital's Rule}} \overset{0/0}{=} \underbrace{\lim_{x\to 0} \frac{e^x}{2}}_{\text{L'Hôpital's Rule}} = \boxed{\frac{1}{2}}$$

7. (10 points) For each part of this problem, find the function $f$ satisfying the given conditions.

(a) $f'(x) = 10x^3 - 9x^2 + 4x, \qquad f(1) = 8$

$$f(x) = \frac{10}{4}x^4 - \frac{9}{3}x^3 + \frac{4}{2}x^2 + C = \frac{5}{2}x^4 - 3x^3 + 2x^2 + C$$

$$f(1) = 8 \Rightarrow \frac{5}{2} - 3 + 2 + C = 8$$

$$C = 9 - \frac{5}{2} = \frac{13}{2}$$

$$\boxed{f(x) = \frac{5}{2}x^4 - 3x^3 + 2x^2 + \frac{13}{2}}$$

(b) $f'(x) = \sin(2x), \qquad f(\pi/4) = -2$

$$f(x) = -\frac{1}{2}\cos(2x) + C$$

$$f(\pi/4) = -2 \Rightarrow -\frac{1}{2}\cos(\pi/2) + C = -2$$

$$\Rightarrow 0 + C = -2 \quad \Rightarrow \quad C = -2$$

$$\boxed{f(x) = -\frac{1}{2}\cos(2x) - 2}$$

8. (10 points) Use 6 subintervals of equal length and subinterval right endpoints to compute a Riemann sum that approximates the value of the integral shown below.

$$\Delta x = \frac{3-1}{6} = \frac{1}{3} \qquad \int_1^3 \frac{2}{x^2}\,dx$$

PARTITION:

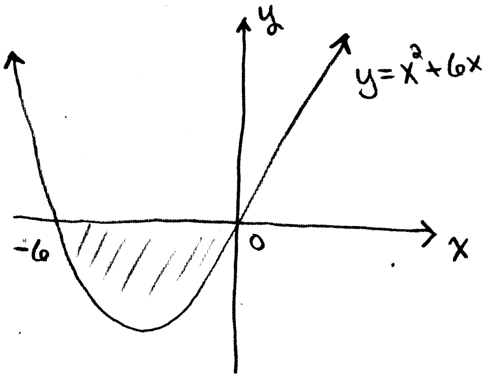
$$1 < \frac{4}{3} < \frac{5}{3} < 2 < \frac{7}{3} < \frac{8}{3} < 3$$

$$\uparrow c_1 \quad \uparrow c_2 \quad \uparrow c_3 \quad \uparrow c_4 \quad \uparrow c_5 \quad \uparrow c_6$$

$$\text{RIEMANN SUM} = \frac{1}{3}\left(\frac{2}{16/9} + \frac{2}{25/9} + \frac{2}{36/9} + \frac{2}{49/9} + \frac{2}{64/9} + \frac{2}{81/9}\right)$$

$$= 6\left(\frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \frac{1}{81}\right) = \frac{1134541}{1058400} \approx 1.07194$$

9. (10 points) Use a definite integral to find the area of the bounded region between the graph of $y = x^2 + 6x$ and the $x$-axis. (It might help to sketch a good graph of the region.)




$x(x+6)=0$

$\Rightarrow x=0,\ x=-6$

$$\int_{-6}^{0} (x^2+6x)\,dx = \frac{1}{3}x^3+3x^2 \Big|_{-6}^{0}$$

$$= 0-(-72+108) = -36$$

NEGATIVE BECAUSE REGION IS BELOW X-AXIS. $\boxed{\text{AREA} = 36 \text{ units}^2}$

10. (10 points) Evaluate each integral. (You may need to use a substitution.)

(a) $\int \left(\frac{1}{x} + \csc^2 x + \frac{1}{1+x^2}\right)\,dx$

$$= \boxed{\ln|x| - \cot x + \tan^{-1} x + C}$$

(b) $\int_1^3 \frac{x^2+1}{\sqrt{x^3+3x}}\,dx$

$$= \frac{1}{3}\int_4^{36} u^{-1/2}\,du = \frac{1}{3}\left(2u^{1/2}\right)\Big|_4^{36} = \frac{2}{3}\sqrt{u}\,\Big|_4^{36}$$

$u = x^3+3x$

$du = (3x^2+3)\,dx$

$\frac{1}{3}du = (x^2+1)\,dx$

$x=1 \Rightarrow u=4$

$x=3 \Rightarrow u=36$

$$\frac{2}{3}\sqrt{36} - \frac{2}{3}\sqrt{4}$$

$$= \boxed{\frac{8}{3}}$$