The integration by parts formula

By now you are certainly aware of the fact that for every basic differentiation rule, there is a related antidifferentiation, or integration, rule. Just to refresh your memory, consider these familiar examples:

$$\frac{d}{dx}\frac{x^{n+1}}{n+1} = x^n \qquad \Longrightarrow \qquad \int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\frac{d}{dx}\sin x = \cos x \qquad \Longrightarrow \qquad \int \cos x\,dx = \sin x + C$$

In this section you will be introduced to the integration rule that is related to the product rule. This new rule, or technique, is called *integration by parts*.

Recall that the product rule states

$$\frac{d}{dx}f(x)g(x) = f(x)g'(x) + f'(x)g(x).$$

After integrating both sides, we obtain

$$f(x)g(x) = \int f(x)g'(x)\,dx + \int f'(x)g(x)\,dx.$$

This is the integration by parts formula, but most often it is written in the following form:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx. \tag{1}$$

Many of you will prefer to remember the more compact form obtained by substituting $u = f(x)$ and $v = g(x)$:

$$\int u\,dv = uv - \int v\,du. \tag{2}$$

Although the need for the integration by parts formula may not be immediately obvious, we will see that it is remarkably useful when evaluating integrals such as

$$\int xe^x\,dx, \quad \int x^3\ln x\,dx, \quad \text{and} \int \sec^3 x\,dx.$$

The integration by parts formula is a reduction formula. After applying it, we will still have another integral to evaluate. Our hope is that the remaining integral is in some sense easier than the original.

In order to integrate by parts, you must first choose $u = f(x)$ and $dv = g'(x)\,dx$. For beginners, this choice is often difficult. u must be easy to differentiate (which by now is the case for most elementary functions), and v must be easily obtained from dv. As if that's not enough to keep in mind, the product $v\,du$ must also be easy to integrate, or at the very least, no more difficult than $u\,dv$.

Let's take a look at a first example. Given enough time, you could probably guess the following integral, but it is a very typical integration by parts problem.

Example 1 Use integration by parts to evaluate $\int x \sin x\, dx$.

There are two obvious choices for u and dv: $u = x$, $dv = \sin x\, dx$ or $u = \sin x$, $dv = x\, dx$. Which choice should we make? With either choice, the functions are easy to differentiate and integrate. So which choice gives a simpler $v\,du$? Let's try them both.

$$u = x,\ dv = \sin x\, dx \qquad \Longrightarrow \qquad du = dx,\ v = -\cos x$$

$$u = \sin x,\ dv = x\, dx \qquad \Longrightarrow \qquad du = \cos x\, dx,\ v = \frac{1}{2}x^2$$

While either choice could be used in the integration by parts formula, only the first choice gives a simpler $v\,du$. It would be easier to work with this choice.

$$\int x \sin x\, dx = -x\cos x - \int -\cos x\, dx$$

$$\int x \sin x\, dx = -x\cos x + \int \cos x\, dx$$

$$\int x \sin x\, dx = -x\cos x + \sin x + C$$

In the example above, the constants of integration were ignored until the very end. We'll always do this. Just don't forget to include a constant at the end.

Example 2 Try this one on your own. Integrate: $\int xe^x\, dx$.

Choosing u and dv

Choosing u and dv wisely gets easier with experience. Until you gain that experience, the acronym LIATE may help you.

LIATE is derived from the first letters of the five basic types of functions you will encounter in an integral: **L**ogarithmic, **I**nverse trigonometric, **A**lgebraic,

Trigonometric, **E**xponential. As a general rule of thumb, choose u in (2) according to LIATE[1]. That is, choose u as the type of function that appears first in LIATE. For example, if an integrand involves the product of an exponential function and a trigonometric function, we would choose u to the the trig function.

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Choose u according to LIATE.

Example 3 Use integration by parts to evaluate $\displaystyle\int_1^2 x^3 \ln x\,dx$.

We'll first evaluate the indefinite integral $\int x^3 \ln x\,dx$, and then we'll tackle the bounds. According to LIATE, we choose $u = \ln x$ and $dv = x^3\,dx$. It follows that

$$du = \frac{1}{x}\,dx, \quad v = \frac{1}{4}x^4.$$

So, according to the integration by parts formula,

$$\begin{aligned}\int x^3 \ln x\,dx &= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^3\,dx \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C\end{aligned}$$

Therefore,

$$\begin{aligned}\int_1^2 x^3 \ln x\,dx &= \frac{1}{4}x^4 \ln x\,\Big|_1^2 - \frac{1}{16}x^4\,\Big|_1^2 \\ &= (4\ln 2 - 0) - \left(1 - \frac{1}{16}\right)\end{aligned}$$

[1]Some people prefer to use LIPTE for **L**ogarithmic, **I**nverse trig, **P**olynomial, **T**rig, **E**xponential.

Based on the previous example, this next statement is probably obvious. Nonetheless, it's worthwhile to make the point.

Integration by Parts - Definite Integrals

$$\int_{x=a}^{x=b} u\,dv = uv\,\Big|_{x=a}^{x=b} - \int_{x=a}^{x=b} v\,du$$

Example 4 Try this one on your own: Find the area of the region bounded above by the graph of $f(x) = (x^2 - 1)\ln x$ and below by the x-axis over the interval $1 \le x \le 3$.

Repeated Integration by Parts

Occasionally an integration problem may require two or more applications of the integration by parts formula. A typical example is the following.

Example 5 Integrate: $\displaystyle\int x^2 e^{2x}\,dx$.

According to LIATE, we choose $u = x^2$ and $dv = e^{2x}\,dx$. With this choice we have

$$du = 2x\,dx, \quad v = \frac{1}{2}e^{2x}$$

and our integral becomes

$$\int x^2 e^{2x}\,dx = \frac{1}{2}x^2 e^{2x} - \int x e^{2x}\,dx.$$

The integral that remains requires integration by parts. Focusing on that integral, we choose $u = x$ and $dv = e^{2x}\,dx$. This gives

$$du = dx, \quad v = \frac{1}{2}e^{2x}.$$

Being very careful with the subtraction, we now have

$$\int x^2 e^{2x}\,dx = \frac{1}{2}x^2 e^{2x} - \frac{1}{2}x e^{2x} + \int \frac{1}{2}e^{2x}\,dx.$$

And finally,

$$\int x^2 e^{2x}\,dx = \frac{1}{2}x^2 e^{2x} - \frac{1}{2}x e^{2x} + \frac{1}{4}e^{2x} + C.$$

Example 6 Try this one on your own. Integrate: $\displaystyle\int x^2 \cos x \, dx.$

Example 7 Rotate the 1st quadrant region bounded by the graphs of $y = \ln x$, $y = 0$, $x = 1$, and $x = 2$ about the x-axis. Use the disk method to find the volume of the solid that is generated.

We start by sketching a graph of the bounded region.

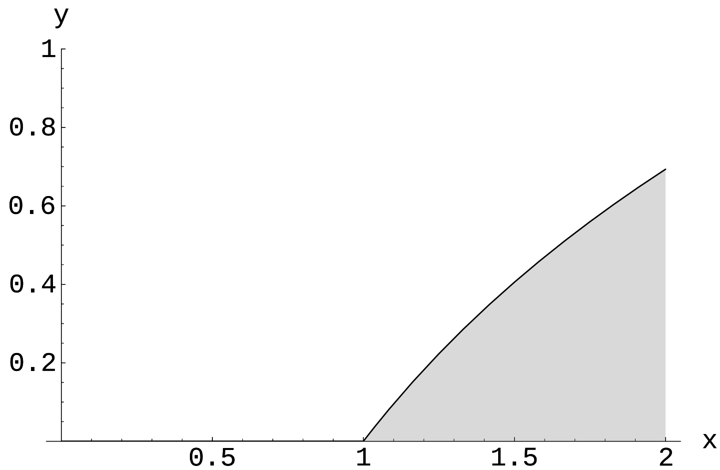

Figure 1: Graph for Example 7.

Using the disk method, we find that the volume is given by

$$V = \pi \int_1^2 (\ln x)^2 \, dx.$$

Although its far from obvious, this integral is best tackled by parts. According to LIATE, we choose $u = (\ln x)^2$ and $dv = dx$. This gives

$$du = 2(\ln x)\frac{1}{x}\, dx, \qquad v = x$$

and

$$V = \pi \int_1^2 (\ln x)^2 \, dx = \pi x(\ln x)^2 \Big|_1^2 - 2\pi \int_1^2 \ln x \, dx.$$

The rest is left as an exercise. You should get

$$V = 2\pi \left[(\ln 2)^2 - 2\ln 2 + 1\right] \text{units}^3 \approx 0.591616 \, \text{units}^3.$$

Recurring Integrals

When integrating by parts, it is not uncommon for the original integrand to show up for a second time. The next example is the first to illustrate this point.

Example 8 Use integration by parts to evaluate $\int \sin x \, \cos x \, dx$.

Typically this problem would be handled by a simple substitution, but we'll use parts. You may be tempted to use LIATE and choose $u = 1$ and $dv = \sin x \, \cos x \, dx$. In doing so, notice that we would need to evaluate the original integral to find v. This can't be right, so let's choose $u = \sin x$ and $dv = \cos x \, dx$. It follows that $du = \cos x \, dx$ and $v = \sin x$ and, after applying the integration by parts formula, we obtain

$$\int \sin x \, \cos x \, dx = \sin^2 x - \int \sin x \, \cos x \, dx.$$

The original integral has reappeared! If we continue to integrate by parts, it will continue to reappear. However, if we get clever and simply add the integral to both sides of the equation, we get

$$2 \int \sin x \, \cos x \, dx = \sin^2 x + C.$$

In past examples we have introduced the constant C after the final integration. In this case, we have gotten around doing a final integration. Our integration problem is solved after dividing by 2:

$$\int \sin x \, \cos x \, dx = \frac{1}{2} \sin^2 x + C.$$

Example 9 Integrate: $\int \sec^3 x \, dx$.

As in Example 8, LIATE is not very helpful. However, a moment's consideration might lead us to try $u = \sec x$ and $dv = \sec^2 x \, dx$. This gives $du = \sec x \tan x \, dx$, $v = \tan x$, and

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx.$$

Using the Pythagorean identity $1 + \tan^2 x = \sec^2 x$, we obtain

$$\int \sec^3 x \, dx = \sec x \tan x - \int (\sec^3 x - \sec x) \, dx$$

or

$$\begin{aligned} 2\int \sec^3 x\,dx &= \sec x \tan x + \int \sec x\,dx \\ &= \sec x \tan x + \ln|\sec x + \tan x| + C. \end{aligned}$$

Our final result is

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

Example 10 Try this one on your own. Integrate: $\int e^x \cos x\,dx$. (Hint: Integrate by parts twice, using LIATE each time. Watch for a recurring integral!)

The Tabular Method

When repeated applications of the integration by parts formula are required, it is often convenient to organize the computations in a table. This is illustrated below where the indefinite integral $\int x^3 \cos x\,dx$ is evaluated.

signs	u and du/dx	dv/dx and $\int dv$
$+$	x^3	$\cos x$
$-$	$3x^2$	$\sin x$
$+$	$6x$	$-\cos x$
$-$	6	$-\sin x$
$+$	0	$\cos x$

$$\begin{aligned} \int x^3 \cos x\,dx &= x^3 \sin x - 3x^2(-\cos x) + 6x(-\sin x) - 6\cos x + \int 0 \cdot \cos x\,dx \\ &= x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6\cos x + C \end{aligned}$$

The first column in the table contains alternating signs starting with $+$, the second column contains u and subsequent derivatives, and the third contains dv/dx and subsequent antiderivatives. We read the table across from the sign and then diagonally downward from u to v. Each repetition of this pattern represents one application of the integration by parts formula (well, technically, one-half an application). Notice that in the example above, we eventually reached a zero derivative in the middle column. This certainly indicates an end to the process, but we could stop at any point as illustrated here:

$$
\begin{tikzcd}
\underline{\text{signs}} & \underline{u \text{ and } du/dx} & \underline{dv/dx \text{ and } \int dv} \\
+ \arrow[r] & x^3 \arrow[dr] & \cos x \\
- \arrow[r] & 3x^2 \arrow[dr] & \sin x \\
+ \arrow[r] & 6x \arrow[r] & -\cos x
\end{tikzcd}
$$

We conclude that

$$\int x^3 \cos x \, dx = x^3 \sin x + 3x^2 \cos x - \int 6x \cos x \, dx.$$

Example 11 Try this one on your own: Use the tabular method to evaluate $\displaystyle\int e^x \cos x \, dx$. Stop when when table has three rows. Compare your results with those of Example 10.

Fourier Analysis

The tabular method of integration by parts works especially well for integrals of the form $\displaystyle\int x^n \sin x \, dx$, $\displaystyle\int x^n \cos x \, dx$, or $\displaystyle\int x^n e^x \, dx$. These types of integrals frequently arise in an application called *Fourier analysis*. Without going into great detail, here is the main idea:

Theorem 1 *Suppose f is a continuous function on the interval $[-\ell, \ell]$. Then f can be approximated on $(-\ell, \ell)$ by the sum*

$$\frac{a_0}{2} + \sum_{n=1}^{M} a_n \cos \frac{n\pi x}{\ell} + \sum_{n=1}^{M} b_n \sin \frac{n\pi x}{\ell} \tag{3}$$

where a_n and b_n are called Fourier coefficients and are given by

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell}\, dx \quad \textit{and} \quad b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell}\, dx.$$

Furthermore, the approximation gets better as M increases.

Example 12 Find the Fourier coefficients of the function $f(x) = x^3$ on the interval $(-1, 1)$.

Let's start by computing the a_n's. For $n = 0, 1, 2, 3, \ldots$, a_n is given by the definite integral

$$a_n = \int_{-1}^{1} x^3 \cos n\pi x\, dx.$$

For any whole number n, the integrand is an odd function. Therefore, we can save some work and immediately conclude that each a_n is zero.

We now focus on b_n's. For $n = 1, 2, 3, \ldots$, b_n is given by the definite integral

$$\int_{-1}^{1} x^3 \sin n\pi x\, dx.$$

For any natural number n, the integrand is an even function. Therefore

$$b_n = \int_{-1}^{1} x^3 \sin n\pi x\, dx = 2 \int_{0}^{1} x^3 \sin n\pi x\, dx.$$

We can now use the tabular method of integration by parts to evaluate the integral and determine b_n.

signs	u and du/dx	dv/dx and $\int dv$
$+$	x^3	$\sin n\pi x$
$-$	$3x^2$	$\frac{-1}{n\pi}\cos n\pi x$
$+$	$6x$	$\frac{-1}{n^2\pi^2}\sin n\pi x$
$-$	6	$\frac{1}{n^3\pi^3}\cos n\pi x$
$+$	0	$\frac{1}{n^4\pi^4}\sin n\pi x$

$$b_n = 2\left(-\frac{x^3}{n\pi}\cos n\pi x + \frac{3x^2}{n^2\pi^2}\sin n\pi x + \frac{6x}{n^3\pi^3}\cos n\pi x - \frac{6}{n^4\pi^4}\sin n\pi x\right)\Bigg|_0^1$$

$$b_n = 2\left(\frac{-1}{n\pi}\cos n\pi + \frac{6}{n^3\pi^3}\cos n\pi\right) = \frac{12-2n^2\pi^2}{n^3\pi^3}(-1)^n$$

Now according to Theorem 1, the following approximation is valid on the interval from $x=-1$ to $x=1$:

$$x^3 \approx \sum_{n=1}^{M} \frac{12-2n^2\pi^2}{n^3\pi^3}(-1)^n \sin n\pi x.$$

In addition, the approximation gets better as M increases.

Shown below are the graphs of $f(x) = x^3$ and the approximating sums for $M=5$ and $M=10$, respectively.

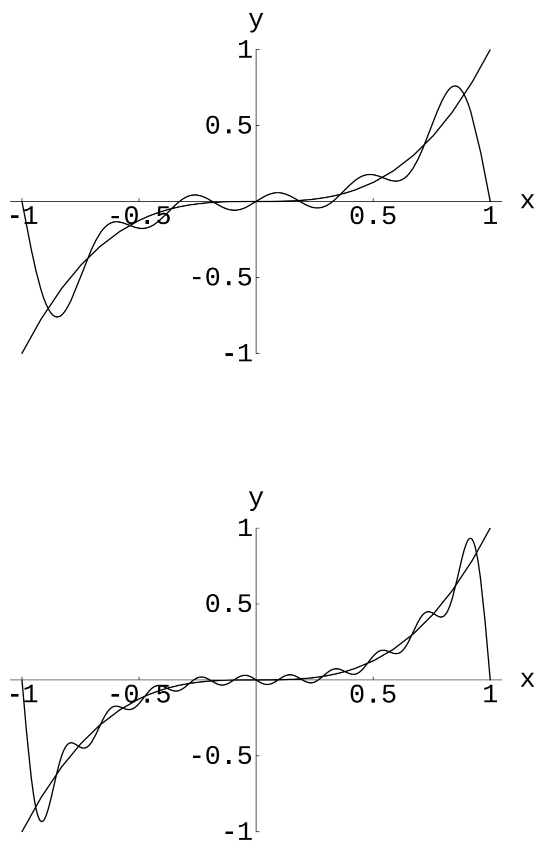

Figure 2: Graphs of $y = x^3$ and the approximating sums from Theorem 1 with $M = 5$ and $M = 10$.

Example 13 Try this one on your own: Find the Fourier coefficients of the function $g(x) = x^2, -\pi < x < \pi$.

Taylor Polynomials

Early in Calculus I you encountered tangent lines and linearizations. Recall that the linearization of f at c is defined by

$$L(x) = f(c) + f'(c)(x - c).$$

L is simply the linear function whose graph is tangent to that of $y = f(x)$ at the point $(c, f(c))$. In practice, $L(x)$ often provides a reasonable approximation for $f(x)$ if x is close to c.

The idea of the linearization is generalized in the following theorem. Its proof involves an interesting theoretical application of repeated integration by parts.

Exercises

Routine Problems

Choose u and dv, determine the corresponding v and du, and write a sentence explaining why you think your choice was a good one.

1. $\displaystyle\int x \ln x \, dx$
2. $\displaystyle\int x^2 \sin 3x \, dx$
3. $\displaystyle\int x \sin^{-1} x^2 \, dx$
4. $\displaystyle\int \ln x \, dx$
5. $\displaystyle\int \tan^{-1} 2x \, dx$
6. $\displaystyle\int e^{2x} \sin 3x \, dx$

Evaluate each integral.

7. $\displaystyle\int \sin^{-1} x \, dx$
8. $\displaystyle\int \frac{\ln x}{x^2} \, dx$

Non-Typical Integration by Parts Problems

Each of the following integrals can be evaluated without resorting to integration by parts. Oh well! Use integration by parts anyway. By careful, LIATE may not apply.

9. $\displaystyle\int x\sqrt{x-1} \, dx$
10. $\displaystyle\int \cos x \; \sin x \; dx$

Recurring Integrals

In each of the following problems be on the lookout for recurring integrals.

11. $\int e^{2x} \sin 3x \, dx$

12. $\int \cos 2x \, \sin 5x \, dx$

13. $\int sec^5 x \, dx$

Miscellaneous Problems

14. Evaluate the indefinite integral $\int (\sin^{-1} x)^2 \, dx$.

15. The bounded region in Example 4 is rotated about the y-axis to form a solid. Find the volume of that solid.

16. The bounded region in Example 4 is rotated about the line $x = 1$ to form a solid. Find the volume of that solid.

17. Consider the indefinite integral $\int \sin ax \, \cos bx \, dx$, where a and b are nonzero real numbers with $a^2 \neq b^2$.

    (a) Use integration by parts to evaluate the integral.

    (b) Use the appropriate trigonometric product-to-sum formula to simplify the integrand. Then evaluate.

    (c) Use a CAS (computer algebra system) to evaluate the integral.

18. (a) Use integration by parts to evaluate $\int \sin^{-1} x \, dx$. (b) Explain how the figure below can be used to arrive at the same conclusion.