Section 5.5 - Alternating Series

Up to now, much of our attention has been focused on series with positive terms. See, for example, the integral test and the comparison tests. In this section, we will begin a take a closer look at more general series.

Definition 1

An alternating series is a series whose terms consistently alternate in sign. Every alternating series can be written in the form

$$\sum(-1)^n a_n \qquad \text{or} \qquad \sum(-1)^{n+1} a_n,$$

where $a_n > 0$. (Notice how we are separating the sign from a_n. We haven't done that before.)

Examples 1

Here are some alternating series...

A. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

B. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} = -1 + \frac{1}{8} - \frac{1}{27} + \frac{1}{64} - \frac{1}{125} + \cdots$

C. $\displaystyle\sum_{n=0}^{\infty} \cos(\pi n) = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$

Here are some series that are NOT alternating series...

D. $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ E. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)/2}}{3^n} = \frac{1}{3} - \frac{1}{9} + \frac{1}{27} + \frac{1}{81} - \frac{1}{243} - \cdots$

Theorem 1 (Alternating series test)

The alternating series of the form

$$\sum(-1)^n a_n \qquad \text{or} \qquad \sum(-1)^{n+1} a_n$$

converges if

1. $0 < a_{n+1} \leq a_n$ for all sufficiently large n, and
2. $\lim\limits_{n\to\infty} a_n = 0$.

That is, if $\{a_n\}$ is eventually non-increasing and has limit zero, then the alternating series converges.

Example 2

Consider the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

In this case $a_n = \frac{1}{n}$. It is pretty clear that each of these values is positive with

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

and

$$\frac{1}{n+1} < \frac{1}{n}$$

for all n. Therefore, by the alternating series test, the series converges. ♢

Example 3

Consider the series $\sum_{n=0}^{\infty} \frac{n+1}{(-2)^n}$.

In this example $a_n = \frac{n+1}{2^n}$. Let $f(x) = \frac{x+1}{2^x}$.

$$f'(x) = \frac{1-(\ln 2)(x+1)}{2^x},$$

which is negative for $x > \frac{1}{\ln 2} - 1$. So f is eventually decreasing, and using L'Hopital's rule,

$$\lim_{x\to\infty} \frac{x+1}{2^x} = \lim_{x\to\infty} \frac{1}{2^x \ln 2} = 0.$$

Therefore a_n is decreasing for $n \geq 1$ and approaching zero. By the alternating series test, the series converges. ♢

Definition 2

Suppose $\sum a_n$ is an infinite series with any combination of positive or negative terms. If the series of absolute values, $\sum |a_n|$, converges, the original series is said to be absolutely convergent.

Example 4

Consider the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

The series of absolute values is $\sum_{n=1}^{\infty} \frac{1}{n^2}$. This new series is a convergent p-series ($p = 2$). So the original series is absolutely convergent. ♢

Theorem 2

If $\sum |a_n|$ converges, then $\sum a_n$ converges. That is, if a series converges absolutely, then it converges in its original form.

Example 5

Let's look back at example 1E: $\sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)/2}}{3^n}$.

The numerator of each term is either $+1$ or -1, so the series of absolute values is $\sum_{n=1}^{\infty} \frac{1}{3^n}$. This new series is a convergent geometric series ($r = 1/3$). Therefore the original series converges absolutely. Since it converges absolutely, it converges. ♢

Example 6

Consider the series $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\sqrt{k}}$.

It is easy to verify that the series converges by the alternating series test. However, the series of absolute values, $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$, is a divergent p-series ($p = 1/2$). The original series is convergent, but not absolutely convergent. ♢

Definition 3

If the series $\sum a_n$ converges, but does not converge absolutely, then the series is said to converge conditionally.

Example 7

Every alternating p-series, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$, with $0 < p \leq 1$, is conditionally convergent.

Comments

1. The ideas of absolute and conditional convergence only apply to series with both positive and negative terms.
2. If a series with positive terms converges, it is automatically converges absolutely. In such a case, there is no need to distinguish between convergence and absolute convergence.
3. A series with positive terms cannot converge conditionally. In such a case conditional convergence makes no sense.

Theorem 3 (Interesting theorem due to Riemann)

1. The terms of an absolutely convergent series can be rearranged in any order with no effect on its convergence or its sum. (Order does not matter for absolutely convergent series.)
2. The terms of a conditionally convergent series can be rearranged in such a way that its sum is any real number. (Order matters for conditionally convergent series.)

Theorem 4 (Alternating series remainder)

Suppose the series $\sum (-1)^n a_n$ or $\sum (-1)^{n+1} a_n$ converges with sum S and satisfies the conditions of the alternating series test. Let S_n denote the nth partial sum of the series. Then

$$|S - S_n| < a_{n+1}.$$

This theorem tells us that the error made in using a partial sum to estimate the sum of a convergent alternating series is bounded by the first neglected term.

Example 8

Look back at example 2. The alternating harmonic series, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, converges by the alternating series test. Whatever value the sum has, let's call it S. If we use first nine terms of the series to estimate S, we get

$$S \approx S_9 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} = \frac{1879}{2520} \approx 0.745634920634920634 9$$

According to theorem 4,

$$|S - S_9| < \frac{1}{10},$$

so that the approximation is off by no more than 0.1. $\diamondsuit$

Example 9

Recall that $n!$ means $1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$. Approximate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$ so that the error in the approximation is less than 0.0005.

This series is an alternating series with $a_n = \frac{1}{n!}$. The a_n's are pretty clearing positive, decreasing, and tending to zero. The conditions of the alternating series test are met. We'd like to find the number n so that

$$|S - S_n| < 0.005.$$

According to theorem 4, this inequality will be satisfied as long as we have

$$a_{n+1} = \frac{1}{(n+1)!} < 0.005.$$

Let's just guess and check...

$$\begin{gathered}
n = 1 \Longrightarrow \frac{1}{2!} = 0.5 \\
n = 2 \Longrightarrow \frac{1}{3!} = 0.166667 \\
n = 3 \Longrightarrow \frac{1}{4!} = 0.0416667 \\
n = 4 \Longrightarrow \frac{1}{5!} = 0.00833333 \\
n = 5 \Longrightarrow \frac{1}{6!} = 0.00138889
\end{gathered}$$

Therefore, $n = 5$ will give the required accuracy.

$$S \approx \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} = 0.633333,$$

and this approximation differs from the exact value of the series by less than 0.005. $\diamondsuit$