Lecture 37 - Integrals Resulting in Exponential, Logarithmic, or Inverse Trigonometric Functions

Objectives

1. Evaluate integrals involving exponential functions.
2. Evaluate integrals that result in logarithmic functions.
3. Evaluate integrals that result in inverse trigonometric functions.

In this lecture, we will work through a number of examples of integration where the results involve exponential, logarithmic, or inverse trigonometric functions. We will use the following integration formulas, which are easy to verify by differentiating the right-hand sides.

Some integration rules

1. $$\int e^x \, dx = e^x + C$$
2. $$\int a^x \, dx = \frac{1}{\ln a} a^x + C$$
3. $$\int \frac{1}{x} \, dx = \ln |x| + C$$
4. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C, \qquad a > 0$$
5. $$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C, \qquad a > 0$$
6. $$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} + C, \qquad a > 0$$

Example 1

Evaluate the indefinite integral $\displaystyle\int e^{-x} \, dx$.

This integral is very close to #1, but it requires a u-substitution.

Let $u = -x$, so that

$$du = -1 \, dx \qquad \text{or} \qquad -du = dx.$$

Upon substituting, we have

$$\int e^{-x} \, dx = -\int e^u \, du.$$

Now we can use #1 to obtain

$$-\int e^u \, du = -e^u + C = -e^{-x} + C. \qquad \diamondsuit$$

Example 2

Evaluate the indefinite integral $\int 5^x \, dx$.

This is precisely a #2 with $a = 5$.

$$\int 5^x \, dx = \frac{1}{\ln 5} 5^x + C \qquad \diamondsuit$$

Example 3

Evaluate the indefinite integral $\int e^x \sqrt{1 + e^x} \, dx$.

First rewrite

$$\int e^x \sqrt{1 + e^x} \, dx = \int e^x (1 + e^x)^{1/2} \, dx.$$

Now the integral is set up nicely for a u-substitution. Let $u = 1 + e^x$, so that

$$du = (0 + e^x) \, dx \qquad \text{or} \qquad du = e^x \, dx.$$

Upon substituting, we have

$$\int e^x (1 + e^x)^{1/2} \, dx = \int u^{1/2} \, du.$$

The new integral is easily evaluated by using the power rule:

$$\int u^{1/2} \, du = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (1 + e^x)^{3/2} + C. \qquad \diamondsuit$$

Example 4

Evaluate the indefinite integral $\int 2x^3 e^{x^4} \, dx$.

Again, this is a nice integral for a u-substitution.

Let $u = x^4$, so that

$$du = 4x^3 \, dx \qquad \text{or} \qquad \frac{1}{4} du = x^3 dx.$$

Upon substituting, we have

$$\int 2x^3 e^{x^4} \, dx = \frac{2}{4} \int e^u \, du.$$

Now we can use #1 to obtain

$$\frac{1}{2} \int e^u \, du = \frac{1}{2} e^u + C = \frac{1}{2} e^{x^4} + C. \qquad \diamondsuit$$

Example 5

Evaluate the definite integral $\int_1^2 \frac{e^{1/x}}{x^2} \, dx$.

Let $u = 1/x$, so that

$$du = -\frac{1}{x^2}\,dx \qquad \text{or} \qquad -\,du = \frac{1}{x^2}\,dx$$
$$x = 1 \Longrightarrow u = 1$$
$$x = 2 \Longrightarrow u = 1/2.$$

Upon substituting, we have

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx = -\int_1^{1/2} e^u\,du = \int_{1/2}^1 e^u\,du.$$

The integral on the right is easy to evaluate, and since we changed the bounds with our substitution, there is no need to resubstitute.

$$\int_{1/2}^1 e^u\,du = e^u\Big|_{1/2}^1 = e - \sqrt{e} \approx 1.06956 \qquad \diamondsuit$$

Example 6

Evaluate the indefinite integral $\int \frac{7}{x}\,dx$.

Even though the integrand could be written as $7x^{-1}$, we cannot use the power rule antidifferentiate. We must recognize that this is a #3.

$$\int \frac{7}{x}\,dx = 7\int \frac{1}{x}\,dx = 7\ln|x| + C \qquad \diamondsuit$$

Example 7

Evaluate the indefinite integral $\int \frac{3}{x-5}\,dx$.

In order to use #3, we must do a u-substitution.

Let $u = x - 5$, so that

$$du = dx.$$

Substitutions in which $du = dx$ are fairly trivial, and with practice and experience, you'll learn to get along without them. For now, let's substitute and evaluate...

$$\int \frac{3}{x-5}\,dx = 3\int \frac{1}{u}\,du = 3\ln|u| + C = 3\ln|x-5| + C \qquad \diamondsuit$$

Example 8

Evaluate the indefinite integral $\int \frac{1-x^2}{3x-x^3}\,dx$ by using the substitution $u = 3x - x^3$.

Let $u = 3x - x^3$, so that

$$du = (3 - 3x^2)\,dx \qquad \text{or} \qquad \frac{1}{3}\,du = (1 - x^2)\,dx.$$

Upon substituting, we have

$$\int \frac{1-x^2}{3x-x^3}\,dx = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|3x - x^3| + C. \qquad \diamondsuit$$

Example 9

Evaluate the definite integral $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$.

This is precisely a #4 with $a = 1$:

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x\Big|_0^1 = \sin^{-1}(1) - \sin^{-1}(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}. \qquad \diamondsuit$$

Example 10

Evaluate the indefinite integral $\displaystyle\int \frac{dx}{\sqrt{4-9x^2}}$.

Let $u = 3x$, so that

$$du = 3\,dx \qquad \text{or} \qquad \frac{1}{3}\,du = dx.$$

Upon substituting, we have

$$\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\int \frac{du}{\sqrt{4-u^2}}.$$

This is now a #4 with $a = 2$:

$$\frac{1}{3}\int \frac{du}{\sqrt{4-u^2}} = \frac{1}{3}\sin^{-1}\frac{u}{2} + C = \frac{1}{3}\sin^{-1}\frac{3x}{2} + C. \qquad \diamondsuit$$

Example 11

Evaluate the indefinite integral $\displaystyle\int \frac{dx}{16+x^2}$.

This is precisely a #5 with $a = 4$:

$$\int \frac{dx}{16+x^2} = \frac{1}{4}\tan^{-1}\frac{x}{4} + C. \qquad \diamondsuit$$