Lecture 27: Optimization

Objectives:

(27.1) Use calculus techniques to solve application problems involving optimization.

Optimization

Optimization problems are application problems that require us to find the maximum or minimum values of a function. These kinds of problems are among the most important of applied mathematics.

We will use the following steps to solve optimization problems.

Step 1 - Identify all given information and all information to be determined. Name and define all necessary variables. Sketch a picture or diagram (if appropriate).

Step 2 - Determine the objective function (i.e the function to be maximized or minimized).

Step 3 - Determine the constraint equation(s), if any.

Step 4 - Use the constraint equation(s) to reduce the objective function to a single-variable function.

Step 5 - Determine the domain of the single-variable objective function.

Step 6 - Use calculus techniques to find the desired maximum or minimum values.

Not every one of these steps will be required for every optimization problem, Nonetheless, the steps provide a pretty thorough framework, and we should think them through with each problem.

Example 1 Find two nonnegative numbers whose sum is 20 and whose product is as great as possible.

Let x and y represent the two nonnegative numbers. Our objective is to maximize the product $P = xy$ subject to the constraint $x + y = 20$. We must first reduce the two-variable objective function, $P = xy$, to a function of a single variable.

$$x + y = 20 \Longrightarrow y = 20 - x$$

$$P = xy \Longrightarrow P(x) = x(20 - x) = 20x - x^2$$

As a polynomial function, P is defined for all real numbers. However, in the context of this particular problem, x must be between 0 and 20 (inclusive). Our goal, then, is to find the maximum value of $P(x) = 20x - x^2$ on the closed and bounded interval $[0, 20]$. We use the techniques of Lecture 22.

$$P'(x) = 20 - 2x = 2(10 - x)$$

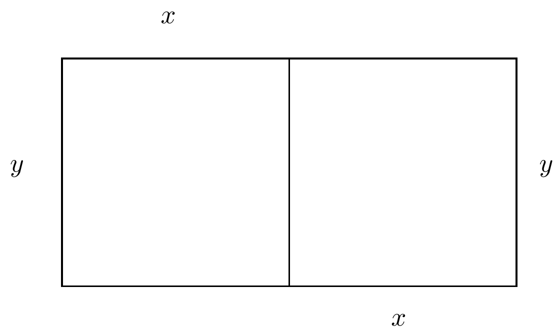
The only critical number of P is $x = 10$. Next we evaluate P at the critical number and the domain endpoints.

x	0	10	20
$P(x)$	0	100	0

$P(x)$ is a maximum when $x = 10$, and the maximum value is $P(10) = 100$. The two nonnegative numbers we're looking for are $x = 10$ and $y = 20 - 10 = 10$.

Example 2 A farmer intends to construct a rectangular pen that will be divided down the middle into two equal-sized pens. If the farmer has 500 ft of fencing material, find the dimensions of the rectangular pen that will have maximum area.

Let x and y represent the length and width of each of the smaller sections of the rectangular pen (see figure below).



Our objective is to maximize the area of the pen, $A = 2xy$, subject to the constraint that the perimeter is 500 ft, $4x + 3y = 500$. We now use the constraint equation to reduce the objective function to a single-variable function.

$$4x + 3y = 500 \Longrightarrow y = \frac{500 - 4x}{3}$$

$$A = 2xy \Longrightarrow A(x) = 2x \cdot \left(\frac{500 - 4x}{3}\right) = \frac{1000}{3}x - \frac{8}{3}x^2$$

In the context of the problem, we must have $0 < x < 125$. So our task is to find the maximum value of $A(x) = \frac{1000}{3}x - \frac{8}{3}x^2$ on the interval $(0, 125)$.

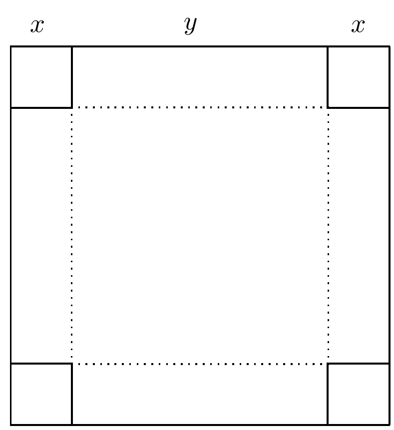
$$A'(x) = \frac{1000}{3} - \frac{16}{3}x = 0 \Longrightarrow x = \frac{1000}{16} = 62.5$$

The only critical number is $x = 62.5$. Since $A''(x) = -\frac{16}{3}$, the graph of A is concave up on $(0, 125)$. Therefore $x = 62.5$ gives us our required maximum. The dimensions that maximize the area are

$$x = 62.5\,\text{ft} \quad \text{and} \quad y = \frac{500 - 4(62.5)}{3} = 83.\overline{3}\,\text{ft}.$$

Example 3 Equal-sized squares will be cut from the corners of a 12 in by 12 in piece of sheet metal. The sides will then be turned up to form an open-top box. Find the dimensions of the box with the greatest volume.

Let x be the length and width of the square cut from each corner. Let y represent the remaining length and width along each side (see figure below).



Once the corners are removed and the sides are folded up, the volume of the box will be $V = xy^2$. Our problem is to maximize $V = xy^2$ subject to $2x + y = 12$.

$$2x + y = 12 \Longrightarrow x = \frac{12 - y}{2}$$

$$V = xy^2 \Longrightarrow V(y) = \left(\frac{12-y}{2}\right)y^2 = 6y^2 - \frac{1}{2}y^3$$

The feasible domain of the volume function is $0 \leq y \leq 12$. Our task is to maximize $V(y)$ on the closed and bounded interval $[0, 12]$.

$$V'(y) = 12y - \frac{3}{2}y^2 = 0 \Longrightarrow y = 0,\ y = 8$$

We now evaluate V at critical numbers and domain endpoints.

y	0	8	12
$V(y)$	0	128	0

The maximum volume occurs when $y = 8$. This makes the height of the box equal to $x = \dfrac{12-8}{2} = 2$. Therefore the dimensions of the box of maximum volume are

$$8\,\text{in} \times 8\,\text{in} \times 2\,\text{in}.$$

Example 4 A manufacturer is designing a 1000 cm^3 can that has the shape of a closed right circular cylinder. What dimensions will produce a can with the minimum surface area?

Let r and h represent the radius and height of the can, respectively. The objective is to minimize the surface area (including the top and bottom),

$$S = 2\pi r^2 + 2\pi rh$$

subject to the volume being 1000,

$$\pi r^2 h = 1000.$$

The details are omitted, but we should find that $r \approx 5.42\,\text{cm}$ and $h \approx 10.84\,\text{cm}$.