Lecture 14: Rates of change and higher-order derivatives

Objectives:

(14.1) Interpret the derivative as a rate of change.

(14.2) Solve falling-body problems.

(14.3) Evaluate higher-order derivatives.

Rates of change

In algebra we learned that the slope of a line could be interpreted as a rate of change. For example, if an auto mechanic charges \$75 to diagnose a problem and then \$50 per hour for the repair, the mechanic will be paid $c(t) = 50t + 75$ dollars for t hours of work. The graph of c is a line with slope 50, and the slope is the per-hour rate at which the mechanic is paid. Remembering that derivatives measure slopes of tangent lines, it is not surprising that derivatives can be interpreted as rates of change.

In the auto mechanic scenario, the rate of change is constant—the mechanic is always paid \$50 per hour. In many situations, however, we shouldn't expect rates of change to be constant. For example, populations tend to grow faster as they get bigger, bank accounts grow faster as they grow, the radius of a balloon grows slower as it inflates, etc. We should only expect a constant rate of change for a linear function. This is completely in line with our knowledge of derivatives.

Example 1 A square has area given by $A(s) = s^2\,\text{cm}^2$, where s is the side length measured in centimeters. Find and interpret $A'(s)$.

$A'(s) = 2s$. A' is the rate of change of area with respect to side length. At the instant that the side length is s, the area is changing at a rate of $2s\,\text{cm}^2$ per $1\,\text{cm}$ change in side length.

Example 2 An object begins falling at $t = 0$ under the influence of gravity only. Its height at time t (in seconds) is given by

$$s(t) = -16t^2 + 320,$$

where s is measured in feet.

1. What is the height at the moment the object begins to fall?

   The object begins to fall at $t = 0$. The initial height is $s(0) = 320\,\text{ft}$.

2. How far does the object fall in its first 4 seconds?

   $s(4) = -16(16)+320 = 64\,\text{ft}$ and $s(0) = 320\,\text{ft}$. The change in height is $s(4)-s(0) = -256\,\text{ft}$. The object falls $256\,\text{ft}$.

3. What is the object's average velocity over the first 4 seconds?

   Average velocity is change in position divided by change in time.

   $$\frac{\Delta s}{\Delta t} = \frac{-256}{4} = -64\,\text{ft/sec}$$

   The average velocity is negative because the height is decreasing, i.e. the object is falling.

4. When does the object hit the ground?

   The object hits the ground when $s(t) = 0$.

   $$s(t) = 0 \Longrightarrow 16t^2 = 320 \Longrightarrow t^2 = 20 \Longrightarrow t = 2\sqrt{5}\,\text{sec}$$

   Notice that we are not interested in the negative square root.

5. Is the object falling faster at the start of its journey or the end of its journey?

The object picks up speed as it falls. Without air resistance, it will continue to fall faster and faster.

6. Find and interpret $s'(t)$.

   $s'(t) = -32t$. s' is the rate of change of height. In other words, $s'(t)$ gives the object's velocity at time t. In this example, the velocity is measured in feet per second.

7. What do $s'(0)$ and $s'(2\sqrt{5})$ represent?

   $s'(0)$ is the object's initial speed. Since $s'(0) = 0$, this object begins to fall from rest, i.e. it is not thrown, but dropped. $s'(2\sqrt{5})$ is the velocity at which the object will hit the ground. The object will obviously come to rest **after** hitting the ground, but it will be moving quite fast when it actually hits. The height function no longer applies after impact.

8. Find the average speed over the interval from $t = 3.99$ to $t = 4$. Compare this with $s'(4)$. Should they be close? Why?

   Solution omitted.

As we saw in the previous example, the velocity of an object is the rate of change of its position. Using differentiation, we will soon be able to justify the following formula.

Falling Body Formula

The height of an object falling under the influence of gravity only is given by

$$s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0,$$

where s_0 is the object's initial height, v_0 is the initial velocity, and g is the acceleration due to the force of gravity. On earth, $g \approx 32\,\text{ft/sec}^2$ or $g \approx 9.8\,\text{m/sec}^2$.

Example 3 An object is launched upward at a speed of 112 ft/sec from over the side of a cliff of height 128 ft.

1. Find a formula for the height of the object at time t.

   We are obviously working in English units, so we'll use $g = 32$.

   $$s(t) = -16t^2 + 112t + 128,$$

   where s is measured in feet.

2. Find a formula for the velocity at time t.

   The velocity is the rate of change of position, i.e $v(t) = s'(t)$.

   $$v(t) = s'(t) = -32t + 112,$$

   where v is measured in feet per second.

3. When will the object reach its maximum height?

   The object will move upward until it stops and begins its downward journey. The max height will occur when the object stops, i.e. when its velocity is zero.

   $$v(t) = 0 \Longrightarrow -32t + 112 = 0 \Longrightarrow 32t = 112 \Longrightarrow t = 3.5\,\text{sec}$$

4. What is the object's maximum height?

   $s(3.5) = -16(3.5)^2 + 112(3.5) + 128 = 324$. The max height is 324 ft.

5. When does the object hit the ground at the bottom of the cliff?

   The object hits the ground when $s(t) = 0$.

$$-16t^2 + 112t + 128 = 0 \Longrightarrow -16(t-8)(t+1) = 0 \Longrightarrow t = 8$$

   The object hits the ground after 8 seconds.

6. What are the velocity and speed when the object hits the ground?

   The velocity is $s'(8) = -32(8) + 112 = -144\,\text{ft/sec}$. The speed is $144\,\text{ft/sec}$.

7. Find and interpret $v'(t)$.

   $v'(t) = -32$. $v'(t)$ is the rate of change of velocity. It measures how fast the velocity changes. We call this *acceleration*.

Higher-order derivatives

In the last part of the example above, we computed the derivative of the velocity function. Notice that $v'(t)$ is really just the second derivative of s: $v'(t) = s''(t)$.

If $s(t)$ is an object's position function, then $s'(t)$ is the velocity function, and $s''(t)$ is the acceleration function.

Summary of position, velocity, & acceleration

If a particle's position at time t is given by $s(t)$, then

- velocity $= v(t) = s'(t)$
- speed $= |v(t)|$
- acceleration $= a(t) = v'(t) = s''(t)$
- average velocity over $[t, t+\Delta t] = \dfrac{s(t+\Delta t) - s(t)}{\Delta t}$

We will see that 2nd- and higher-order derivatives have many practical applications. To compute higher-order derivatives, we simply differentiate repeatedly. The notation for higher-order derivatives is similar to that used for 1st derivatives.

1st derivative	$f'(x)$	$\dfrac{dy}{dx}$
2nd derivative	$f''(x)$	$\dfrac{d^2y}{dx^2}$
3rd derivative	$f'''(x)$	$\dfrac{d^3y}{dx^3}$
4th derivative	$f^{(4)}(x)$	$\dfrac{d^4y}{dx^4}$
$\vdots$	$\vdots$	$\vdots$

Example 4 Find $p'(x)$ and $p''(x)$ if $p(x) = 8x^3 + 5x^2 - 17$.

$$p'(x) = 24x^2 + 10x \qquad \text{and} \qquad p''(x) = 48x + 10$$

Example 5 Let $f(x) = x^2 \sin x$. Find $f''(x)$.

$$f'(x) = x^2 \cos x + 2x \sin x$$

$$f''(x) = -x^2 \sin x + 2x \cos x + 2x \cos x + 2 \sin x = -x^2 \sin x + 4x \cos x + 2 \sin x$$

Example 6 Find $\dfrac{d^{12}y}{dx^{12}}$ if $y = 5x^7 + \sin x$.

Instead of taking 12 derivatives, let's use reason. First, by using our basic differentiation rules, we can take the 12th derivatives of $5x^7$ and $\sin x$ separately, and then add the results. To differentiate $5x^7$, we could use the power rule repeatedly, each application decreasing the exponent by 1. However, after 8 derivatives, we'll get the zero function. Now if we think about the higher-order derivatives of $y = \sin x$:

$$\frac{dy}{dx} = \cos x, \qquad \frac{d^2y}{dx^2} = -\sin x, \qquad \frac{d^3y}{dx^3} = -\cos x, \qquad \frac{d^4y}{dx^4} = \sin x,$$

we see that the derivatives of orders 4, 8, 12, etc. must be $\sin x$. When all is said and done, we have

$$\frac{d^{12}y}{dx^{12}} = 0 + \sin x.$$