Lecture 13: Product rule and quotient rule

Objectives:

(13.1) Use the product and quotient rules to evaluate derivatives.

(13.2) Differentiate the six basic trigonometric functions.

Product and quotient rules

The rules for differentiating products and quotients of functions are a little more complicated than most students first expect. In order to master these rules, we will have to do lots of practice problems. When practicing, it may be worthwhile to use a computer algebra system (CAS) to check our work. A free, easy-to-use, online CAS is Wolfram Alpha, available at `http://www.wolframalpha.com/`. The syntax is simple—to compute the derivative of $5x^2 + 3x$, simply enter `derivative of 5x^2+3x`. Wolfram Alpha will even show the steps!

Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

A proof of the product rule is given at the end of this set of lecture notes. For now, let's focus on some examples.

Example 1 Find the derivative of $h(x) = (5x^2 + 3x)(8x^3 - x^2)$.

According to the product rule,

$$h'(x) = (5x^2 + 3x)\frac{d}{dx}(8x^3 - x^2) + (8x^3 - x^2)\frac{d}{dx}(5x^2 + 3x)$$

or

$$h'(x) = (5x^2 + 3x)(24x^2 - 2x) + (8x^3 - x^2)(10x + 3).$$

In general, we will not simplify our derivatives unless there is a compelling reason to do so. We can check our work using Wolfram Alpha by entering `derivative of (5x^2+3x)*(8x^3-x^2)`.

Example 2 Let $y = 5x^2 \cos x$. Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = (5x^2)(-\sin x) + (\cos x)(10x) = 10x\cos x - 5x^2 \sin x$$

The Wolfram Alpha syntax is: `derivative of 5x^2*cos(x)`.

Example 3 Evaluate the derivative: $\frac{d}{dt}[\sqrt{t}\sin t]$.

We start by writing $\sqrt{t} = t^{1/2}$. Then use the product rule.

$$\frac{d}{dt}[t^{1/2}\sin t] = t^{1/2}\cos t + (\sin t)(\frac{1}{2}t^{-1/2}) = \sqrt{t}\cos t + \frac{\sin t}{2\sqrt{t}}.$$

The Wolfram Alpha syntax is: `derivative of sqrt(t)*sin(t)`.

Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \text{ provided } g(x) \neq 0$$

A mnemonic device that may be helpful for remembering the quotient rule is the following:

"Low D High minus High D Low all over Low squared"

where Low and High refer to the numerator and denominator, respectively, and D means to differentiate.

Example 4 Let $y = \dfrac{x^2-4}{5x+7}$. Find $\dfrac{dy}{dx}$.

According to the quotient rule,

$$\frac{dy}{dx} = \frac{(5x+7)\frac{d}{dx}(x^2-4) - (x^2-4)\frac{d}{dx}(5x+7)}{(5x+7)^2} = \frac{(5x+7)(2x) - (x^2-4)(5)}{(5x+7)^2} = \frac{5x^2+14x+20}{(5x+7)^2}.$$

The Wolfram Alpha syntax is: `derivative of (x^2-4)/(5x+7)`.

Example 5 Use the quotient rule to differentiate $y = \tan x$.

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Now that we've learned how to differentiate quotients, we can evaluate the derivatives of all of the trigonometric functions. The results are summarized in the following table.

Derivatives of the trigonometric functions

$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\cos x = -\sin x$
$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\cot x = -\csc^2 x$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\frac{d}{dx}\csc x = -\csc x \cot x$

Example 6 Let $g(x) = \dfrac{x^2 \sec x}{x^3+1}$. Find $g'(x)$.

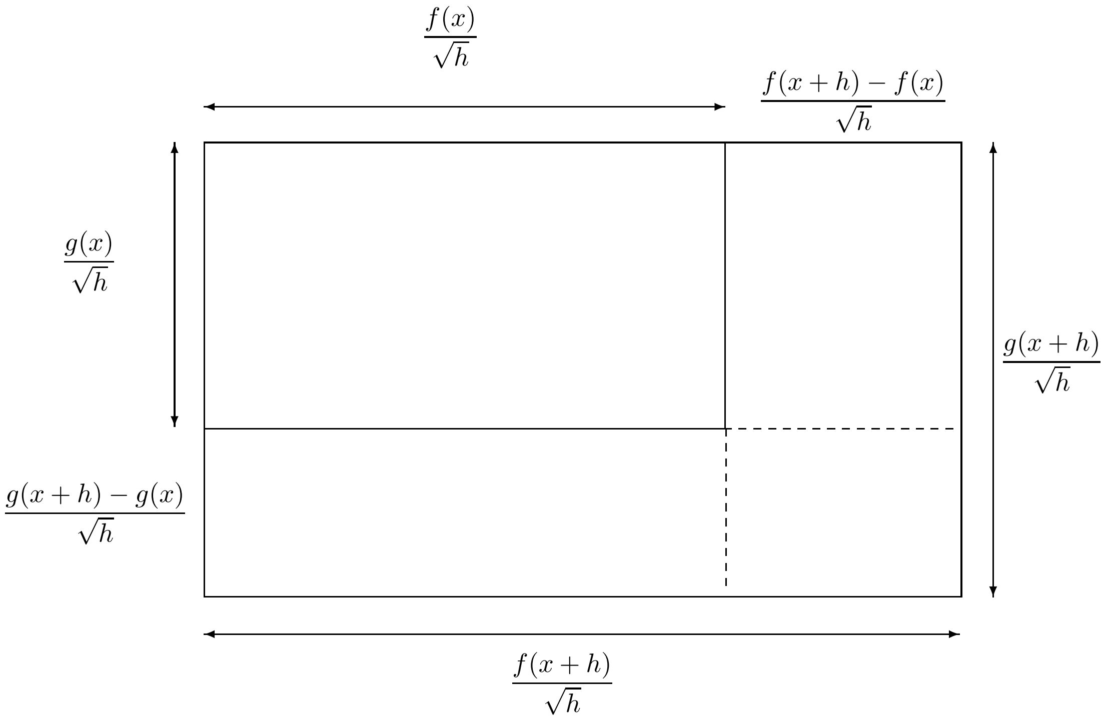
We need both the product and quotient rules.

$$g'(x) = \frac{(x^3+1)(2x\sec x + x^2 \sec x \tan x) - (x^2 \sec x)(3x^2)}{(x^3+1)^2}$$

The Wolfram Alpha syntax is: `derivative of (x^2*sec(x))/(x^3+1)`.

Justification for product rule

Consider a rectangle that measures $f(x+h)/\sqrt{h}$ by $g(x+h)/\sqrt{h}$ and an embedded rectangle that measures $f(x)/\sqrt{h}$ by $g(x)/\sqrt{h}$.



The area of the region between the rectangles is given by

$$\frac{f(x+h)g(x+h)-f(x)g(x)}{h} = \left(\frac{f(x+h)-f(x)}{h}\right)g(x)+\left(\frac{g(x+h)-g(x)}{h}\right)f(x)+[g(x+h)-g(x)]\left(\frac{f(x+h)-f(x)}{h}\right).$$

Now we take the limit as $h \to 0$ to get

$$\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} = f'(x)g(x)+g'(x)f(x)+(0)f'(x)$$

or

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x)+g'(x)f(x).$$

Justification for quotient rule

If we assume that $h(x)=\dfrac{f(x)}{g(x)}$ is differentiable, it is easy to use the product rule to derive the quotient rule. We multiply by $g(x)$, differentiate, and solve for $h'(x)$:

$$g(x)h(x)=f(x) \Longrightarrow g(x)h'(x)+h(x)g'(x)=f'(x) \Longrightarrow h'(x)=\frac{f'(x)-h(x)g'(x)}{g(x)}$$

After substituting $h(x)=f(x)/g(x)$, we have the quotient rule.