Lecture 1: Review: Lines, equations, and rationalizing

Objectives:

(1.1) Use points and slope to find equations of lines.

(1.2) Solve basic equations with and without technology.

(1.3) Rationalize numerators and denominators of expressions.

Lines and slope

For any numbers A, B, and C (with A and B not both zero), the set of all points (x, y) satisfying $Ax+By = C$ is a line in the xy-plane. The standard form, $Ax + By = C$, is not unique, but the line determined by A, B, and C is.

In geometry, you learned that every line is uniquely determined by two distinct points that lie on it. In algebra and calculus, it is more convenient to think of the defining characteristics as the direction of the line (characterized by slope or lack thereof) and a single point on the line.

The slope of a nonvertical line is given by its ratio of rise to run (relative to the positive x-axis):

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

for any distinct points (x_1, y_1) and (x_2, y_2) on the line. The slope of a vertical line is not defined.

Example 1 Suppose (x_1, y_1) and (x_2, y_2) satisfy $Ax + By = C$. Find the slope of line in terms of A, B, and C.

Since the two points lie on the line described by $Ax + By = C$, we must have $Ax_1 + By_1 = C$ and $Ax_2 + By_2 = C$. It follows that

$$(Ax_2 + By_2) - (Ax_1 + By_1) = 0 \qquad \text{or} \qquad A(x_2 - x_1) + B(y_2 - y_1) = 0.$$

Therefore

$$A(x_2 - x_1) = -B(y_2 - y_1) \qquad \text{or} \qquad \frac{A}{-B} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Example 2 What can be said about A, B, and C if the line described by $Ax + By = C$ happens to be vertical?

A vertical line has the same, constant x-value for any y-value. Therefore, the equation of a vertical line cannot depend on the variable y. We must have $B = 0$. Any vertical line has an equation that can be written in the form $x = a$.

For a variety of reasons, we will mostly be interested in nonvertical lines. The standard form equation of a nonvertical line can be rewritten in many special ways:

(1) Point-slope form: $y - y_0 = m(x - x_0)$

(2) Slope-intercept form: $y = mx + b$

(3) Modified point-slope form: $y = y_0 + m(x - x_0)$

(4) Intercept-intercept form: $\dfrac{x}{a} + \dfrac{y}{b} = 1$

Example 3 A line passes through the points $(4, 3)$ and $(-2, -6)$. Write the equation of the line in each of the forms above and also in standard form.

Solution omitted.

Basic equation solving

Throughout the study of calculus, we will make use of many basic equation solving strategies. We should be prepared to use the quadratic formula, factoring techniques (including factoring by grouping), and basic trig identities. The following examples review some of these strategies.

Example 4 Solve for x: $x^2 = 4 - x^2$

$$x^2 = 4 - x^2 \quad \Longrightarrow \quad 2x^2 = 4 \quad \Longrightarrow \quad x^2 = 2$$
$$x = \sqrt{2} \quad \text{or} \quad x = -\sqrt{2}$$

Example 5 Solve for x: $x^2 - 4x + 3 = -x^2 + 2x + 3$

$$x^2 - 4x + 3 = -x^2 + 2x + 3 \quad \Longrightarrow \quad 2x^2 - 6x = 0 \quad \Longrightarrow \quad 2x(x-3) = 0$$
$$x = 0 \quad \text{or} \quad x = 3$$

Example 6 Solve for x: $(x^2 - 4)^2(-10) - (-10x)(2)(x^2 - 4)(2x) = 0$

$$(x^2 - 4)^2(-10) - (-10x)(2)(x^2 - 4)(2x) = 0 \quad \Longrightarrow \quad -10(x^2 - 4)(x^2 - 4 - 4x^2) = 0$$
$$10(x^2 - 4)(3x^2 + 4) = 0$$
$$x = 2 \quad \text{or} \quad x = -2$$

Example 7 Solve for x: $1 - \dfrac{2}{x^{1/3}} = 0$

$$1 - \frac{2}{x^{1/3}} = 0 \quad \Longrightarrow \quad x^{1/3} = 2 \quad \Longrightarrow \quad x = 8$$

Example 8 Solve for x: $2\cos x + 2\sin 2x = 0$

$$2\cos x + 2\sin 2x = 0 \quad \Longrightarrow \quad 2\cos x + 4\sin x \cos x = 0 \quad \Longrightarrow \quad (2\cos x)(1 + 2\sin x) = 0$$
$$\cos x = 0 \quad \text{or} \quad \sin x = -\frac{1}{2}$$
$$x = \frac{\pi}{2} + 2k\pi,\ -\frac{\pi}{2} + 2k\pi \quad \text{or} \quad x = -\frac{\pi}{6} + 2k\pi,\ -\frac{5\pi}{6} + 2k\pi$$

Example 9 Use your calculator to find the two intersection points of the graphs of $f(x) = \cos x$ and $g(x) = x^2$.

Rounding to the nearest thousandth, the intersection points are $(0.8241, 0.6792)$ and $(-0.8241, 0.6792)$.

Rationalizing numerators and denominators

We will often come across problems where we will be required to rationalize numerators or denominators of rather complicated expressions. In order to do so, we will usually "multiply by the conjugate."

The conjugate of the expression $A+B$ is the expression $A-B$. When using FOIL to multiply conjugates, the inside and outside products will always cancel:

$$(A+B)(A-B) = A^2 - AB + AB - B^2.$$

When rationalizing numerators or denominators, it is usually a good idea to carry out the multiplication only where you expect cancellation. The following examples illustrate this.

Example 10 Rationalize the denominator: $\dfrac{x-9}{\sqrt{x}-3}$

The conjugate of $\sqrt{x}-3$ is $\sqrt{x}+3$, so we multiply both the numerator and the denominator by $\sqrt{x}+3$.

$$\frac{(x-9)(\sqrt{x}+3)}{(\sqrt{x}-3)(\sqrt{x}+3)} = \frac{(x-9)(\sqrt{x}+3)}{x-9} = \sqrt{x}+3$$

Notice that we did not actually carry out the multiplication in the numerator.

Example 11 Rationalize the numerator: $\dfrac{\sqrt{1-x}-2}{x+3}$

The conjugate of $\sqrt{1-x}-2$ is $\sqrt{1-x}+2$.

$$\frac{(\sqrt{1-x}-2)(\sqrt{1-x}+2)}{(x+3)(\sqrt{1-x}+2)} = \frac{(1-x)-4}{(x+3)(\sqrt{1-x}+2)} = \frac{(-x-3)}{(x+3)(\sqrt{1-x}+2)} = \frac{-1}{\sqrt{1-x}+2}$$

Notice that we did not actually carry out the multiplication in the denominator.