Solving First-Order, Linear ODEs

It is common in mathematics to multiply both sides of an equation by an expression that will in some way simplify the equation. In solving a first-order, linear ODE,

$$y'(x) + p(x)y(x) = q(x),$$

we will use the same idea. We seek an *integrating factor*, $\mu(x)$, such that

$$(\mu\, y)' = \mu\, q.$$

Let's see what $\mu(x)$ should look like...

We start with

$$y' + p\, y = q.$$

Multiply both sides by $\mu(x)$:

$$\mu\, y' + \mu\, p\, y = \mu\, q.$$

We want the left-hand side to be equal to $(\mu\, y)'$:

$$(\mu\, y)' = \mu\, y' + \mu'\, y = \mu\, y' + \mu\, p\, y.$$

It follows that

$$\mu'\, y = \mu\, p\, y \quad \text{or} \quad \mu' = \mu\, p.$$

And it is easy to see that the last equation is satisfied by

$$\mu(x) = e^{\int p(x)\, dx}.$$

Solving 1st order, linear ODEs

- Given $y'(x) + p(x)\, y(x) = q(x)$
- Find the integrating factor $\mu(x) = e^{\int p(x)\, dx}$
- The solution follows from

$$\mu(x)\, y(x) = \int \mu(x)\, q(x)\, dx$$

(Don't forget your constant of integration!)

- Or, if you are given an initial condition...

$$\mu(x)\, y(x) = \mu(x_0)\, y(x_0) + \int_{x_0}^{x} \mu(t)\, q(t)\, dt$$